

Traces of analytic uniform algebras on subvarieties and test collections

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ABSTRACT

Given a complex domain Ω and analytic functions $\varphi_1, \dots, \varphi_n : \Omega \rightarrow \mathbb{D}$, we find geometric conditions allowing us to conclude that $H^\infty(\Omega)$ is generated by functions of the form $g \circ \varphi_k$, $g \in H^\infty(\mathbb{D})$. This is then applied to the construction of an extension of bounded holomorphic functions on an analytic one-dimensional complex subvariety of the polydisk \mathbb{D}^n to functions in the Schur-Agler algebra of \mathbb{D}^n , with an estimate on the norm of the extension. Our proofs use an extension of the technique of separation of singularities by Havin, Nersessian and Ortega-Cerdá.

1. Introduction

1.1. The statement of main results

This paper is devoted to the problem of extending a bounded analytic function from a subvariety of the polydisk \mathbb{D}^n to a bounded analytic function on the polydisk, as well as a related problem of the generation of algebras. Our main motivations come from operator theory and concern some tests for K -spectrality and complete K -spectrality of a Hilbert space linear operator. These will be treated in a forthcoming article [18].

Let $\Omega \subset \mathbb{C}$ be a domain and $\Phi : \Omega \rightarrow \mathbb{D}^n$ be an analytic function. Its image $\mathcal{V} = \Phi(\Omega)$ is an analytic variety inside \mathbb{D}^n (which may have singular points). We say that a complex function f defined on \mathcal{V} is analytic if, for every point $p \in \mathcal{V}$, there is a neighborhood U of p in \mathbb{C}^n and an analytic function F on U such that $f|(\mathcal{V} \cap U) = F|(\mathcal{V} \cap U)$. We define $H^\infty(\mathcal{V})$ to be the Banach algebra of bounded analytic functions on \mathcal{V} , equipped with the supremum norm.

A fundamental question is whether it is possible to extend a function in $H^\infty(\mathcal{V})$ to a function in $H^\infty(\mathbb{D}^n)$, the Banach algebra of bounded analytic functions on \mathbb{D}^n , also equipped with the supremum norm. Since the restriction map $H^\infty(\mathbb{D}^n) \rightarrow H^\infty(\mathcal{V})$ is a contractive homomorphism, this question asks whether the image of this homomorphism, $H^\infty(\mathbb{D}^n)|\mathcal{V}$, is all of $H^\infty(\mathcal{V})$.

Denote by Φ^* the pullback by Φ ; that is, the map $\Phi^* : H^\infty(\mathbb{D}^n) \rightarrow H^\infty(\Omega)$ defined by $\Phi^*(f) = f \circ \Phi$. If this map is onto, i.e, if $\Phi^*H^\infty(\mathbb{D}^n) = H^\infty(\Omega)$, then every function in $H^\infty(\mathcal{V})$ can be extended to a function in $H^\infty(\mathbb{D}^n)$, because if $f \in H^\infty(\mathcal{V})$, then $f \circ \Phi \in H^\infty(\Omega)$. When Φ^* is onto, we can find an $F \in H^\infty(\mathbb{D}^n)$ such that $f \circ \Phi = \Phi^*F = F \circ \Phi$. This equality implies that $F|_{\mathcal{V}} = f$, so F extends f to $H^\infty(\mathbb{D}^n)$.

We show that one has $\Phi^*H^\infty(\mathbb{D}^n) = H^\infty(\Omega)$ for a class of domains Ω and functions Φ satisfying certain geometric conditions, and such that Φ is injective and Φ' does not vanish (in this case, \mathcal{V} is an analytic variety). If the hypotheses are weaker (in particular, if Φ is injective

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and $\Phi' \neq 0$ only outside a finite subset of Ω), we show that $\Phi^*H^\infty(\mathbb{D}^n)$ is a finite codimensional subalgebra of $H^\infty(\Omega)$. It is easy to see that one cannot get the whole $H^\infty(\Omega)$ algebra in this case. As will be seen however, even under these weaker assumptions, every function in $H^\infty(\mathcal{V})$ can be extended to a function in $H^\infty(\mathbb{D}^n)$.

We also consider other algebras of functions on \mathbb{D}^n besides $H^\infty(\mathbb{D}^n)$. One of these algebras is $\mathcal{SA}(\mathbb{D}^n)$, the Agler algebra of \mathbb{D}^n . It is the Banach algebra of functions analytic on \mathbb{D}^n such that the norm

$$\|f\|_{\mathcal{SA}(\mathbb{D}^n)} \stackrel{\text{def}}{=} \sup_{\substack{\|T_j\| \leq 1 \\ \sigma(T_j) \subset \mathbb{D}}} \|f(T_1, \dots, T_n)\|$$

is finite. Here the supremum is taken over all tuples (T_1, \dots, T_n) of commuting contractions on a Hilbert space such that the spectra $\sigma(T_j)$ are contained in \mathbb{D} ($f(T_1, \dots, T_n)$ is well defined for such tuples). Clearly, $\mathcal{SA}(\mathbb{D}^n)$ is a subset of $H^\infty(\mathbb{D}^n)$ and $\|f\|_{H^\infty(\mathbb{D}^n)} \leq \|f\|_{\mathcal{SA}(\mathbb{D}^n)}$. For $n = 1, 2$, we have the equality $\mathcal{SA}(\mathbb{D}^n) = H^\infty(\mathbb{D}^n)$, and the norms coincide. However, for $n \geq 3$, the norms do not coincide. Also, if $n \geq 3$, it is currently unknown whether or not $\mathcal{SA}(\mathbb{D}^n)$ is a proper subset of $H^\infty(\mathbb{D}^n)$. The unit ball of the Agler algebra is known as the Schur-Agler class. It turns out that it is the proper analog of the unit ball in $H^\infty(\mathbb{D})$ (the so called Schur class) when studying the Pick interpolation problem in \mathbb{D}^n . The Schur-Agler class also has important applications in operator theory and function theory.

We can ask whether every function in $H^\infty(\mathcal{V})$ can be extended to a function in $\mathcal{SA}(\mathbb{D}^n)$ and whether $\Phi^*\mathcal{SA}(\mathbb{D}^n) = H^\infty(\Omega)$. We show that for the class of functions Φ considered in this article, the answer to the first question is affirmative, and the answer to the second question is also affirmative if Φ is injective and Φ' does not vanish.

Another interesting algebra is $H^\infty(\mathcal{K}_\Psi)$. This algebra, extensively studied in [19], is associated with a collection of test functions Ψ . It turns out that if $\Phi = (\varphi_1, \dots, \varphi_n) : \Omega \rightarrow \mathbb{D}^n$ is injective, then $\{\varphi_1, \dots, \varphi_n\}$ is a collection of test functions, which we also denote by Φ , and $H^\infty(\mathcal{K}_\Phi) = \Phi^*\mathcal{SA}(\mathbb{D}^n)$. Therefore, the question of whether $H^\infty(\mathcal{K}_\Phi) = H^\infty(\Omega)$, is a reformulation of the question from the previous paragraph.

If Ω is a nice domain (say with piecewise smooth boundary), and Φ extends by continuity to $\overline{\Omega}$, then we can also consider the algebra $A(\overline{\Omega})$ of functions analytic in Ω and continuous in $\overline{\Omega}$ instead of $H^\infty(\Omega)$. The set $\overline{\mathcal{V}} = \Phi(\overline{\Omega})$ is a bordered analytic variety, and we can consider the algebra $A(\overline{\mathcal{V}})$ of functions analytic in \mathcal{V} and continuous in $\overline{\mathcal{V}}$. The extension problem can also be formulated for these algebras. One can ask whether every function in $A(\overline{\mathcal{V}})$ extends to a function in $A(\overline{\mathbb{D}^n})$, the algebra of functions analytic in \mathbb{D}^n and continuous in $\overline{\mathbb{D}^n}$, or to $\mathcal{SA}_A(\mathbb{D}^n) \stackrel{\text{def}}{=} \mathcal{SA}(\mathbb{D}^n) \cap A(\overline{\mathbb{D}^n})$. Our methods apply to this problem, and so many of our results have two versions: one for algebras of type H^∞ , another for algebras of functions continuous up to the boundary.

Another important algebra for us is \mathcal{H}_Φ , the (not necessarily closed) subalgebra of $H^\infty(\Omega)$ generated by functions of the form $f \circ \varphi_k$, with $f \in H^\infty(\mathbb{D})$, and $k = 1, \dots, n$:

$$\mathcal{H}_\Phi = \left\{ \sum_{j=1}^l \prod_{k=1}^n f_{j,k}(\varphi_k(z)) : l \in \mathbb{N}, f_{j,k} \in H^\infty(\mathbb{D}) \right\}.$$

We have the following algebra inclusions:

$$\mathcal{H}_\Phi \subset \Phi^*\mathcal{SA}(\mathbb{D}^n) \subset \Phi^*H^\infty(\mathbb{D}^n) \subset \Phi^*H^\infty(\mathcal{V}) \subset H^\infty(\Omega). \quad (1.1)$$

The first inclusion follows from the observation that any function on \mathbb{D}^n of the form $f(z_k)$, with $f \in H^\infty(\mathbb{D})$, belongs to $\mathcal{SA}(\mathbb{D}^n)$ by the von Neumann inequality, as do sums of products of such functions since $\mathcal{SA}(\mathbb{D}^n)$ is an algebra. The inclusion $\Phi^*H^\infty(\mathbb{D}^n) \subset \Phi^*H^\infty(\mathcal{V})$ holds since if $F \in H^\infty(\mathbb{D}^n)$, then $F|_{\mathcal{V}} \in H^\infty(\mathcal{V})$ and $\Phi^*F = \Phi^*(F|_{\mathcal{V}})$.

We define \mathcal{A}_Φ to be the (not necessarily closed) subalgebra of $A(\overline{\Omega})$ generated by functions of the form $f \circ \varphi_k$ with $f \in A(\overline{\mathbb{D}})$ and $k = 1, \dots, n$:

$$\mathcal{A}_\Phi = \left\{ \sum_{j=1}^l \prod_{k=1}^n f_{j,k}(\varphi_k(z)) : l \in \mathbb{N}, f_{j,k} \in A(\overline{\mathbb{D}}) \right\}.$$

We likewise have the inclusions

$$\mathcal{A}_\Phi \subset \Phi^* \mathcal{S} \mathcal{A}_A(\mathbb{D}^n) \subset \Phi^* A(\overline{\mathbb{D}}^n) \subset \Phi^* A(\overline{\mathcal{V}}) \subset A(\overline{\Omega}). \quad (1.2)$$

Some useful terminology: by an open circular sector in \mathbb{C} with vertex on a point z_0 , we mean a set of the form

$$\{z \in \mathbb{C} \setminus \{z_0\} : |z - z_0| < r, \alpha < \arg(z - z_0) < \beta\},$$

where $r > 0$, and $\alpha < \beta < \alpha + 2\pi$. A closed analytic arc is understood as the image of the interval $[0, 1]$ by an analytic function, defined and univalent on an open subset of \mathbb{C} containing $[0, 1]$. We denote by $\mathbb{D}_\varepsilon(z_0)$ the open disk centered at z_0 with radius ε .

Let us now introduce the class of functions Φ to be considered in this article.

DEFINITION. Let Ω be a domain whose boundary is a disjoint finite union of piecewise analytic Jordan curves such that the interior angles of the “corners” of $\partial\Omega$ are in $(0, \pi]$. We say that a function $\Phi = (\varphi_1, \dots, \varphi_n) : \overline{\Omega} \rightarrow \overline{\mathbb{D}}^n$ is *admissible* if $\varphi_k \in A(\overline{\Omega})$, for $k = 1, \dots, n$, and there is a collection of sets $\{J_k\}_{k=1}^n$, where each J_k is a finite union of disjoint closed analytic subarcs of $\partial\Omega$, and a constant α , $0 < \alpha \leq 1$, such that the following conditions are satisfied (see Figure 1):

- (a) $\bigcup_{k=1}^n J_k = \partial\Omega$.
- (b) $|\varphi_k| = 1$ in J_k , for $k = 1, \dots, n$.
- (c) For each $k = 1, \dots, n$, there exists an open set $\Omega_k \supset \Omega$ such that the interior of J_k relative to $\partial\Omega$ is contained in Ω_k , φ_k is defined in $\overline{\Omega}_k$, $\varphi_k \in A(\overline{\Omega}_k)$, and φ'_k is of class Hölder α in Ω_k ; i.e.,

$$|\varphi'_k(\zeta) - \varphi'_k(z)| \leq C|\zeta - z|^\alpha, \quad \zeta, z \in \Omega_k.$$

- (d) If z_0 is an endpoint of one of the arcs comprising J_k , then there exists an open circular sector $S_k(z_0)$ with vertex on z_0 and such that $S_k(z_0) \subset \Omega_k$ and $J_k \cap \mathbb{D}_\varepsilon(z_0) \subset S_k(z_0) \cup \{z_0\}$, for some $\varepsilon > 0$. If z_0 is a common endpoint of both one of the arcs comprising J_k one of the arcs comprising J_l , $k \neq l$, then we require $(S_k(z_0) \cap S_l(z_0)) \setminus \overline{\Omega}$ to be nonempty.
- (e) $|\varphi'_k| \geq C > 0$ in J_k , for $k = 1, \dots, n$.
- (f) For each $k = 1, \dots, n$, $\varphi_k|_{J_k}$ is injective and $\varphi_k(J_k) \cap \varphi_k(\partial\Omega \setminus J_k) = \emptyset$.

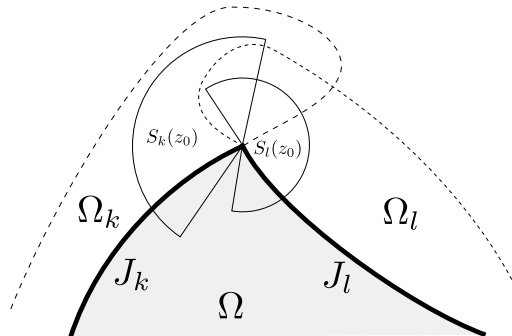


FIGURE 1. The sets involved in the definition of an admissible function

The hypothesis that φ'_k is of class Hölder α in Ω_k can be weakened a little by instead requiring that φ'_k be of class Hölder α only in a relative neighborhood of J_k in $\overline{\Omega}_k$.

It follows from the above hypotheses that if z_0 is an endpoint of one of the arcs comprising J_k , then φ_k is conformal at z_0 . Since $\varphi_k(\overline{\Omega}) \subset \overline{\mathbb{D}}$, and φ_k preserves angles, the interior angle of $\partial\Omega$ at z_0 must be less than or equal than π . This justifies the assumption on the angles at the corners of $\partial\Omega$. This is an important restriction on the class of domains which our methods do not permit us to relax.

By the Schwarz reflection principle and condition (b), one can always find sets Ω_k as in (c) by continuing φ_k analytically across J_k . In general, these sets Ω_k do not intersect in a way that permits the construction of the circular sectors required in (d). However, if all the interior angles of the corners of $\partial\Omega$ are greater than $2\pi/3$, then it is easy to see that Schwarz reflection produces sets Ω_k which contain such circular sectors.

Additionally, if φ_k is defined only in $\overline{\Omega}$, φ'_k is Hölder α on Ω and $|\varphi'_k| \geq C > 0$ in J_k , then the extension of φ_k to Ω_k by Schwarz reflection also satisfies that φ'_k is of class Hölder α .

It is easy to check from the definition of an admissible function Φ that Φ' vanishes at most in a finite number of points in $\overline{\Omega}$ and that there is a finite set $X \subset \Omega$ such that the restriction of Φ to $\overline{\Omega} \setminus X$ is injective (i.e., Φ identifies or “glues” at most a finite number of points of $\overline{\Omega}$).

The main results of this article are the following theorems.

THEOREM 1.1. *If $\Phi : \overline{\Omega} \rightarrow \overline{\mathbb{D}^n}$ is admissible and injective and Φ' does not vanish on Ω , then $\mathcal{H}_\Phi = H^\infty(\Omega)$ and $\mathcal{A}_\Phi = A(\overline{\Omega})$.*

It follows that in this case all the algebras in (1.1) coincide, as do all those in (1.2).

Some of the conditions that we are imposing on Φ are easily seen to in fact be necessary for the equality $\Phi^*H^\infty(\mathbb{D}^n) = H^\infty(\Omega)$, which is weaker than $\Phi^*\mathcal{SA}(\mathbb{D}^n) = H^\infty(\Omega)$, to hold. For instance, if Φ is not injective, then no function in $\Phi^*H^\infty(\mathbb{D}^n)$ is injective, so this set cannot be all of $H^\infty(\Omega)$. Similarly, if $\Phi'(z_0) = 0$ for some $z_0 \in \Omega$, then we have $f'(z_0) = 0$ for every $f \in \Phi^*H^\infty(\mathbb{D}^n)$, which again implies $\Phi^*H^\infty(\mathbb{D}^n) \neq H^\infty(\Omega)$. Finally, if there is a point $z_0 \in \partial\Omega$ such that $|\varphi_k(z_0)| < 1$ for all functions φ_k , then every function in $\Phi^*H^\infty(\mathbb{D}^n)$ is continuous at z_0 , so once again $\Phi^*H^\infty(\mathbb{D}^n) \neq H^\infty(\Omega)$. It is also easy to show that $\Phi^*A(\overline{\mathbb{D}^n}) \neq A(\overline{\Omega})$ in this case as well. This motivates conditions (a) and (b) in the definition of an admissible function.

In the case when Φ is not injective or Φ' vanishes at some points, we prove the following result, which according to the remarks above is the best that we can hope.

LEMMA 1.2. *If $\Phi : \overline{\Omega} \rightarrow \overline{\mathbb{D}^n}$ is admissible, then \mathcal{H}_Φ is a closed subalgebra of finite codimension in $H^\infty(\Omega)$, and \mathcal{A}_Φ is a closed subalgebra of finite codimension in $A(\overline{\Omega})$.*

In fact, we prove below that \mathcal{H}_Φ is also weak*-closed in $H^\infty(\Omega)$ (see Section 5).

Regarding the algebras $H^\infty(\mathcal{V})$ and $A(\overline{\mathcal{V}})$ of functions defined on the analytic curve \mathcal{V} , we prove the following result.

THEOREM 1.3. *If $\Phi : \overline{\Omega} \rightarrow \overline{\mathbb{D}^n}$ is admissible, then $\Phi^*H^\infty(\mathcal{V}) = \mathcal{H}_\Phi$ and $\Phi^*A(\overline{\mathcal{V}}) = \mathcal{A}_\Phi$.*

In this case the first four algebras in (1.1) and the first four in (1.2) coincide, while the last inclusions can be proper, though $\Phi^*H^\infty(\mathcal{V})$ happens to be weak*-closed in $H^\infty(\Omega)$, while $\Phi^*A(\overline{\mathcal{V}})$ is norm closed in $A(\overline{\Omega})$, and both have finite codimension.

This theorem allows us to prove a result on the extension of functions in \mathcal{V} to the Agler algebra.

THEOREM 1.4. *If $\Phi : \overline{\Omega} \rightarrow \overline{\mathbb{D}}^n$ is admissible, for every $f \in H^\infty(\mathcal{V})$ there is an $F \in \mathcal{SA}(\mathbb{D}^n)$ such that $F|_{\mathcal{V}} = f$ and $\|F\|_{\mathcal{SA}(\mathbb{D}^n)} \leq C\|f\|_{H^\infty(\mathcal{V})}$, for some constant C independent of f . Additionally, if $f \in A(\overline{\mathcal{V}})$, then F can be taken to belong to $\mathcal{SA}_A(\mathbb{D}^n)$.*

Our proofs use an extension of the techniques of Havin and Nersessian and Ortega-Cerdá [25, 26], which concern the separation of singularities of bounded analytic functions, defined on open subsets of \mathbb{C} . Havin and Nersessian prove in [25] that if Ω_1, Ω_2 are domains in \mathbb{C} such that their boundaries intersect transversally, then $H^\infty(\Omega_1 \cap \Omega_2) = H^\infty(\Omega_1) + H^\infty(\Omega_2)$, in the sense that every function $f \in H^\infty(\Omega_1)$ can be written as $f = f_1 + f_2$ with $f_j \in H^\infty(\Omega_j)$.

The main tool for the proof of the above theorems is the next, which can be understood as a kind of decomposition result for functions in $H^\infty(\Omega)$. Its proof is given in Section 4.

THEOREM 1.5. *If $\Phi : \overline{\Omega} \rightarrow \overline{\mathbb{D}}^n$ is admissible, then there exist bounded linear operators $F_k : H^\infty(\Omega) \rightarrow H^\infty(\mathbb{D})$, $k = 1, \dots, n$, such that the operator defined by*

$$f \mapsto f - \sum_{k=1}^n F_k(f) \circ \varphi_k, \quad f \in H^\infty(\Omega),$$

is compact in $H^\infty(\Omega)$ and its range is contained in $A(\overline{\Omega})$. Moreover, F_k maps $A(\overline{\Omega})$ into $A(\overline{\mathbb{D}})$, for $k = 1, \dots, n$.

There is some relationship between our setting and the algebra generation problem. Given any finite family $\Phi = \{\varphi_k\} \subset A(\overline{\Omega})$, one can also consider the algebras \overline{A}_Φ , the smallest *norm closed* subalgebra of $A(\overline{\Omega})$ containing Φ , and \overline{H}_Φ^∞ , the weak*-closed subalgebra of $H^\infty(\Omega)$ generated by the family Φ .

PROPOSITION 1.6. *If $\Phi : \overline{\Omega} \rightarrow \overline{\mathbb{D}}^n$ is admissible, then $\mathcal{A}_\Phi = \overline{A}_\Phi$ and $\mathcal{H}_\Phi = \overline{H}_\Phi^\infty$.*

Proof. It is clear that in general, $\mathcal{A}_\Phi \subset \overline{A}_\Phi$ and $\mathcal{H}_\Phi \subset \overline{H}_\Phi^\infty$. By Lemma 1.2, \mathcal{A}_Φ is closed in norm, and by Lemma 5.4, \mathcal{H}_Φ is weak*-closed. The equalities now follow. \square

Theorem 1.1 then implies corresponding results about the generation of algebras.

COROLLARY 1.7. *If $\Phi : \overline{\Omega} \rightarrow \overline{\mathbb{D}}^n$ is admissible and injective and Φ' does not vanish in Ω , then $\overline{A}_\Phi = A(\overline{\Omega})$ and $\overline{H}_\Phi^\infty = H^\infty(\Omega)$.*

The assertions that $\mathcal{A}_\Phi = A(\overline{\Omega})$ and $\mathcal{H}_\Phi = H^\infty(\Omega)$ are much stronger than just the fact that Φ generates algebras $A(\overline{\Omega})$ and $H^\infty(\Omega)$ (in the weak* sense, in the last case). For instance, as was mentioned, the equalities $\mathcal{A}_\Phi = A(\overline{\Omega})$ and $\mathcal{H}_\Phi = H^\infty(\Omega)$ are impossible if there is a point $z_0 \in \partial\Omega$ such that $\max_k |\varphi_k(z_0)| < 1$, whereas Φ still can still generate algebras $A(\overline{\Omega})$ and $H^\infty(\Omega)$ in this case. (Notice that for any nonzero constants $\{\lambda_k\}$, the families $\{\varphi_k\}$ and $\{\lambda_k \varphi_k\}$ generate the same closed subalgebras of $A(\overline{\Omega})$ and $H^\infty(\Omega)$.) In the applications to operator theory that we consider in [18], algebra generation does not suffice, and the assertions that $\mathcal{A}_\Phi = A(\overline{\Omega})$ and $\mathcal{H}_\Phi = H^\infty(\Omega)$ and so Theorem 1.5 play an important role there.

1.2. A brief review of previous results on algebra generation and continuation

The study of generators of algebras of the type $A(\overline{\Omega})$ dates back to Wermer [40], where he considered pairs of functions as generators of the algebra $A(K)$ for a compact subset K of a Riemann surface. Bishop worked on the same problem independently in [9], using a different approach. In these two articles, sufficient conditions are given for generation of the whole algebra, or of a finite codimensional subalgebra. Several later works are devoted to giving weaker sufficient conditions; see [10] and [37]. In [33, 38], the H^p closure rather than of the uniform closure of the algebra is considered, although [38] does also give a result for the disk algebra. Even in the simple case of $A(\overline{\mathbb{D}})$, necessary and sufficient conditions for a pair of functions to generate the whole algebra are still unknown (see [27, Problem 2.32]).

In most articles on algebra generation, it is assumed that the derivatives of the generators are continuous up to the boundary. In our setting, as we remarked after the definition of an admissible function, it is only necessary that each function φ'_k be Hölder continuous near the arc J_k . In this sense, it seems that Corollary 1.7 is a new result. We also stress that our results concern the algebra \mathcal{A}_Φ , which is *a priori* a non-closed algebra smaller than $\overline{\mathcal{A}}_\Phi$, the smallest closed algebra containing $\{\varphi_1, \dots, \varphi_n\}$. In our applications to operator theory in [18], it is essential that the theorems we have stated in the Introduction are proved for \mathcal{A}_Φ rather than its closure.

There is a case in which Theorems 1.1 and 1.5 are a straightforward consequence of the results of Havin and Nersessian [25] on the separation of singularities of analytic functions. In this case a stronger version of Theorem 1.5 can be obtained; namely, one can prove the existence of bounded linear operators $F_k : H^\infty(\Omega) \rightarrow H^\infty(\mathbb{D})$ such that

$$f = \sum_{k=1}^n F_k(f) \circ \varphi_k, \quad f \in H^\infty(\mathbb{D}). \quad (1.3)$$

This case is as follows. Assume that there are simply connected domains D_k , $k = 1, \dots, n$, such that $\Omega = \bigcap_k D_k$, and that φ_k are conformal maps from D_k onto \mathbb{D} . If the boundaries of the domains D_k intersect transversally, by [25, Example 4.1] there are bounded linear operators $G_k : H^\infty(\Omega) \rightarrow H^\infty(D_k)$ such that $f = \sum_k G_k(f)$ for every $f \in H^\infty(\Omega)$. If we put $F_k(f) = G_k(f) \circ \varphi_k^{-1}$, then we get (1.3). From this, the equality $\mathcal{H}_\Phi = H^\infty(\mathbb{D})$ follows trivially. It is not difficult to see that such operators F_k map $A(\overline{\Omega})$ into $A(\overline{\mathbb{D}})$, so we also get the equality $\mathcal{A}_\Phi = A(\overline{\Omega})$.

The case when $\Omega = \mathbb{D}$ is important. Then $\mathcal{V} = \Phi(\mathbb{D})$ is called an analytic disk inside the polydisk. It is a particular kind of hyperbolic analytic curve, the theory of which has been treated extensively in the literature. The function theory of this curves and its relation with finite codimensional subalgebras of holomorphic functions was studied by Agler and McCarthy in [4]. A classification of the finite codimensional subalgebras of a function algebra which is related to the one that we use in the proof of Theorem 1.1 was given by Gamelin in [21].

The problem of extension of a bounded analytic function defined on an analytic curve $\mathcal{V} \subset \mathbb{D}^n$ to the polydisk \mathbb{D}^n dates back to Rudin and to Stout (see [39]), and was also treated by Polyakov in [35], and more generally by Polyakov and Khenkin in [36]. The book [28] by Henkin and Leiterer also treats the extension of bounded analytic functions defined on subvarieties in a fairly general context. In these works, the subvariety \mathcal{V} is assumed to be extendable to a neighborhood of $\overline{\mathbb{D}}^n$, which means that there is a larger analytic subvariety $\tilde{\mathcal{V}}$ of a neighborhood of $\overline{\mathbb{D}}^n$ such that $\mathcal{V} = \tilde{\mathcal{V}} \cap \mathbb{D}^n$. This is in contrast to a function Φ meeting our requirements (a)–(f) “in general position”, in which case the variety $\mathcal{V} = \Phi(\Omega)$ does not extend to a larger analytic variety $\tilde{\mathcal{V}}$.

The works of Amar and Charpentier [8] and of Chee [11, 12] do deal with the setting when this extension of \mathcal{V} may be absent. In [8], extensions by bounded analytic functions to bidisks are considered, whereas the papers [11, 12] concern the case of an analytic variety

\mathcal{V} of codimension 1 in a polydisk \mathbb{D}^n and therefore can be compared with our results only for $n = 2$. Theorem 1.1 in [12] implies that in our setting, for the case of $n = 2$, every $f \in H^\infty(\mathcal{V})$ can be extended to an $F \in H^\infty(\mathbb{D}^2)$ such that $F|_{\mathcal{V}} = f$.

For the case of $\mathcal{V} = \Phi(\mathbb{D})$, where $\Phi : \mathbb{D} \rightarrow \mathbb{D}^n$ extends to a neighborhood of $\overline{\mathbb{D}}$, a necessary and sufficient condition for the property of analytic bounded extension was given by Stout in [39], in which case, at least one φ_k must be a finite Blaschke product.

If $\tilde{\mathcal{V}}$ is an analytic curve in a neighborhood of $\overline{\mathbb{D}}^n$ such that there is a biholomorphic map $\tilde{\Phi}$ of a domain $G \subset \mathbb{C}$ onto $\tilde{\mathcal{V}}$ and $\mathcal{V} = \tilde{\mathcal{V}} \cap \mathbb{D}^n$, the set $\Omega = \tilde{\Phi}^{-1}(\mathcal{V})$ is connected and $\Phi = \tilde{\Phi}|_{\Omega}$, then typically all the above conditions (a)–(e) on Φ are satisfied, whereas (f) is an additional requirement. In this case, if Ω is simply connected and $\hat{\Phi} = \Phi \circ \eta : \mathbb{D} \rightarrow \mathcal{V}$, where η is a Riemann mapping of \mathbb{D} onto Ω , then $\hat{\Phi}$ does not continue analytically to a larger disk unless Ω has analytic boundary. In other words, there are cases when \mathcal{V} has an extension to a larger subvariety whereas Φ does not extend.

Bounded extensions to an analytic polyhedron W in \mathbb{C}^n from a subvariety \mathcal{V} of arbitrary codimension were studied by Adachi, Andersson and Cho in [2]. It was assumed there that \mathcal{V} is continuable to a neighborhood of W . Notice that polydisks are particular cases of analytic polyhedra.

The property of the bounded extension of H^∞ functions does not hold in general, and one can find several counterexamples in the literature, see [6, 16, 17, 34].

There are also many papers in the literature that deal with bounded extensions in the context of strictly pseudoconvex domains or domains with smooth boundary (the polydisk does not belong to these classes). See Diederich and Mazzilli [16, 17] and the recent paper by Alexandre and Mazzilli [5]. Holomorphic extensions have also been studied extensively in different contexts of L^p norms; we refer to Chee [12, 13] for the case of the polydisk. See also the review [1] by Adachi, the paper [5] and references therein for a more complete information.

The above-mentioned papers use diverse techniques from several complex variables, such as the Cousin problem and integral representations for holomorphic functions. In another group of papers, interesting results around the problem of bounded continuation are obtained using the tools of operator theory and the theory of linear systems. Agler and McCarthy [3] treat the bounded extension property with preservation of norms for the bidisk \mathbb{D}^2 . See also [23] for partial results for the case of tridisks and general polydisks. It seems that very few varieties \mathcal{V} have this norm preserving extension property.

In [31], Knešević studies the existence of bounded extensions from distinguished subvarieties of \mathbb{D}^2 (without preservation of norms). His approach is based on certain representations of two-variable transfer functions and permits him to give concrete estimates of the constants.

The same problem can also be studied for the ball \mathbb{B}^n instead of the polydisk. In [7], Alpay, Putinar and Vinnikov use reproducing kernel Hilbert space techniques to show that a bounded analytic function defined on an analytic disk in \mathbb{B}^n can be extended to \mathbb{B}^n . Indeed they show that it can be extended to an element of the multiplier algebra of the Drury-Arveson space $H^2(\mathbb{B}^n)$; this algebra is properly contained in $H^\infty(\mathbb{B}^n)$. See also [15] for further examples and counterexamples, and the relationship with the complete Nevanlinna-Pick property.

The extension problem is also treated in [30], where it appears as a consequence of isomorphism of certain multiplier algebras of analytic varieties. Some problems considered there resemble those we consider on the pullback by Φ .

Our approach differs from the approaches described above, in that it relies on techniques inspired by the Havin-Nersisyan work, certain compactness arguments, which show that some subalgebras of H^∞ have finite codimension, and the study of maximal ideals and derivations in H^∞ . An important aspect distinguishing it from earlier results, is that we can prove continuation to $\mathcal{SA}(\mathbb{D}^n)$. If $\mathcal{SA}(\mathbb{D}^n)$ is strictly contained in $H^\infty(\mathbb{D}^n)$ (it is not known whether this is true), then our results are stronger. Indeed, we prove even more: it follows from the

proof of Theorem 1.4 that there is closed subspace of finite codimension in $H^\infty(\mathcal{V})$ such that every function in this space can be extended to a function of the form $F_1(z_1) + \cdots + F_n(z_n)$, where $F_j \in H^\infty(\mathbb{D})$ for $j = 1, \dots, n$.

Following on from the work in [25], the papers [26] by Havin, Nersessian and Ortega-Cerdà, and [24] by Havin prove separation of singularities under weaker hypothesis. It would be interesting to know if some of these results, in particular the examples at the end of [24] can be used to extend what we do.

1.3. The organization of the paper

Section 2 is devoted to the proof of Theorem 1.1. The proof of this theorem uses Theorem 1.5, which is proved in Section 4. To prove Theorem 1.5, we need to use some facts about weakly singular integral operators, which are given in Section 3. In Section 5, we deal with the weak*-closedness of the algebras that we are treating. In Section 6 we consider finite codimensional subalgebras of a particular kind which we call “glued subalgebras”. These will be a key tool in the sequel. Section 7 contains the proofs of Theorems 1.3 and 1.4. Finally, in Section 8 we give some lemmas regarding families of functions Φ_ε that depend continuously on a parameter ε . These results are not used elsewhere in this article, but they will be essential in [18]. We place them here, because their proof uses the details of the proof of Theorem 1.5.

2. The proof of Theorem 1.1 (modulo Theorem 1.5)

In this section we prove Theorem 1.1. This requires Theorem 1.5, which was stated above and is proved in Section 4. As a first step, we obtain Lemma 1.2, stated in the Introduction, which is a simple consequence of Theorem 1.5.

Proof of Lemma 1.2. By Theorem 1.5 and the standard theory of Fredholm operators, the range of the operator $f \mapsto \sum F_k(f) \circ \psi_k$, $f \in H^\infty(\Omega)$, is a closed subspace of finite codimension in $H^\infty(\Omega)$. Since \mathcal{H}_Φ contains this range, we get that \mathcal{H}_Φ is a closed subalgebra of finite codimension in $H^\infty(\Omega)$. To obtain the analogous result for \mathcal{A}_Φ , we just consider the restriction of the operator $f \mapsto \sum F_k(f) \circ \psi_k$ to $A(\overline{\Omega})$. \square

The proof of Theorem 1.1 follows essentially from Lemma 1.2, together with the application of some Banach algebra techniques. We now recall some basic facts about the maximal ideal space of $H^\infty(\Omega)$ which are used in the proof. We refer to [29, Chapter 10] for a detailed discussion of the Banach algebra structure of $H^\infty(\mathbb{D})$. The properties of $H^\infty(\Omega)$ for a finitely connected domain Ω are similar. We denote by $\mathfrak{M}(H^\infty(\Omega))$ the space of all complex homomorphisms on $H^\infty(\Omega)$, endowed with the weak* topology inherited as a subspace of the dual space $(H^\infty(\Omega))^*$. It is a compact Hausdorff space.

We denote by \mathbf{z} the identity function on Ω , i.e., $\mathbf{z}(z) = z$. For any complex homomorphism $\psi \in \mathfrak{M}(H^\infty(\Omega))$, either $\psi(\mathbf{z}) \in \Omega$ or $\psi(\mathbf{z}) \in \partial\Omega$. If $\psi(\mathbf{z}) = z_0 \in \Omega$, then $\psi(f) = f(z_0)$ for every $f \in H^\infty(\Omega)$. If $\psi(\mathbf{z}) = z_0 \in \partial\Omega$, then we can assert that $\psi(f) = f(z_0)$ for every $f \in H^\infty(\Omega)$ that extends by continuity to z_0 (the proof for $\Omega = \mathbb{D}$, given in [29], easily adapts to any finitely connected domain).

A linear functional $\eta \in (H^\infty(\Omega))^*$, it is called a *derivation* at $\psi \in \mathfrak{M}(H^\infty(\Omega))$ if

$$\eta(fg) = \eta(f)\psi(g) + \psi(f)\eta(g), \quad \forall f, g \in H^\infty(\Omega).$$

It is easy to see that if η is a derivation at ψ with $\psi(\mathbf{z}) = z_0 \in \Omega$, then $\eta(f) = \eta(\mathbf{z})f'(z_0)$ for every $f \in H^\infty(\Omega)$ (one must check first that $\eta(1) = 0$ and then write $f = f(z_0) + f'(z_0)(\mathbf{z} - z_0) + (\mathbf{z} - z_0)^2g$ with $g \in H^\infty(\Omega)$). Derivations at ψ with $\psi(\mathbf{z}) \in \partial\Omega$ have the following somewhat similar property.

LEMMA 2.1. Let $f \in H^\infty(\Omega)$ be continuous at $z_0 \in \partial\Omega$ with $(f - f(z_0))/(\mathbf{z} - z_0) \in H^\infty(\Omega)$. If η is a derivation in $H^\infty(\Omega)$ at $\psi \in \mathfrak{M}(H^\infty(\Omega))$ with $\psi(\mathbf{z}) = z_0$, then

$$\eta(f) = \eta(\mathbf{z})\psi\left(\frac{f - f(z_0)}{\mathbf{z} - z_0}\right).$$

Proof. We just compute

$$\begin{aligned} \eta(f) &= \eta(f - f(z_0)) = \eta\left((\mathbf{z} - z_0)\frac{f - f(z_0)}{\mathbf{z} - z_0}\right) \\ &= \psi(\mathbf{z} - z_0)\eta\left(\frac{f - f(z_0)}{\mathbf{z} - z_0}\right) + \eta(\mathbf{z} - z_0)\psi\left(\frac{f - f(z_0)}{\mathbf{z} - z_0}\right) = \eta(\mathbf{z})\psi\left(\frac{f - f(z_0)}{\mathbf{z} - z_0}\right). \end{aligned}$$

□

LEMMA 2.2. If $\Phi : \overline{\Omega} \rightarrow \overline{\mathbb{D}}^n$ is admissible and $\psi_1 \neq \psi_2$ are in $\mathfrak{M}(H^\infty(\Omega))$ and satisfy $\psi_1(f) = \psi_2(f)$ for every $f \in \mathcal{H}_\Phi$, then $\psi_1(\mathbf{z}), \psi_2(\mathbf{z}) \in \Omega$, $\psi_1(\mathbf{z}) \neq \psi_2(\mathbf{z})$ and $\Phi(\psi_1(\mathbf{z})) = \Phi(\psi_2(\mathbf{z}))$. The same is true if $H^\infty(\Omega)$ is replaced by $A(\overline{\Omega})$ and \mathcal{H}_Φ is replaced by \mathcal{A}_Φ .

Proof. Since by assumption $\varphi_k \in A(\overline{\Omega})$, we have $\psi_j(\varphi_k) = \varphi_k(\psi_j(\mathbf{z}))$ for $j = 1, 2$, $k = 1, \dots, n$. Therefore, $\Phi(\psi_1(\mathbf{z})) = \Phi(\psi_2(\mathbf{z}))$, because the functions φ_k belong to \mathcal{H}_Φ . Let $z_j = \psi_j(\mathbf{z}) \in \overline{\Omega}$. If $z_1 \in \partial\Omega$, then by condition (f), $z_2 = z_1$, and hence $\psi_1(f) = \psi_2(f)$ for all $f \in A(\overline{\Omega})$. Take an $f \in H^\infty(\Omega)$ and put $g = \sum_{k=1}^N F_k(f) \circ \varphi_k$, where F_k are as in Theorem 1.5. Then $f - g \in A(\overline{\Omega})$ and $g \in \mathcal{H}_\Phi$. Therefore, we have $\psi_1(f - g) = \psi_2(f - g)$, and also $\psi_1(g) = \psi_2(g)$. It follows that $\psi_1(f) = \psi_2(f)$, so that $\psi_1 = \psi_2$, because f was arbitrary. This contradicts our assumption. Hence $\psi_j(\mathbf{z}) \in \Omega$ for $j = 1, 2$ and, since $\psi_1 \neq \psi_2$, it must happen that $\psi_1(\mathbf{z}) \neq \psi_2(\mathbf{z})$. The reasoning for $A(\overline{\Omega})$ is the same. □

LEMMA 2.3. If $\Phi : \overline{\Omega} \rightarrow \overline{\mathbb{D}}^n$ is admissible and $\eta \neq 0$ is a derivation of $H^\infty(\Omega)$ at $\psi \in \mathfrak{M}(H^\infty(\Omega))$ such that $\eta(f) = 0$ for every $f \in \mathcal{H}_\Phi$, then $\psi(\mathbf{z}) \in \Omega$ and $\Phi'(\psi(\mathbf{z})) = 0$. The same is true if $H^\infty(\Omega)$ is replaced by $A(\overline{\Omega})$ and \mathcal{H}_Φ is replaced by \mathcal{A}_Φ .

Proof. We consider two cases according to whether $\psi(\mathbf{z})$ belongs to Ω or to $\partial\Omega$. The case when $\psi(\mathbf{z}) \in \Omega$ is clear: since $\varphi_k \in \mathcal{H}_\Phi$, we have $0 = \eta(\varphi_k) = \varphi'_k(\psi(\mathbf{z}))$, and so $\Phi'(\psi(\mathbf{z})) = 0$.

Now we show that the case $\psi(\mathbf{z}) \in \partial\Omega$ cannot happen. Here we also distinguish two cases according to whether $\eta(\mathbf{z})$ is zero or not. If $\eta(\mathbf{z}) \neq 0$, then we take $k \in \{1, \dots, n\}$ such that $\psi(\mathbf{z}) \in J_k$. Since φ_k is derivable at $\psi(\mathbf{z})$, we have $\eta(\varphi_k) = \eta(\mathbf{z})\varphi'_k(\psi(\mathbf{z}))$ by Lemma 2.1. Therefore, $\varphi'_k(\psi(\mathbf{z})) = 0$, because $\varphi_k \in \mathcal{H}_\Phi$. This contradicts condition (e) in the definition of an admissible family.

In the case when $\eta(\mathbf{z}) = 0$, we get $\eta(f) = 0$ for every f analytic on some neighborhood of $\overline{\Omega}$. This implies $\eta(f) = 0$ for every $f \in A(\overline{\Omega})$, because functions analytic on $\overline{\Omega}$ are dense in $A(\overline{\Omega})$. Now take $f \in H^\infty(\Omega)$ and put $g = \sum_{k=1}^n F_k(f) \circ \varphi_k$, where F_k are as in Theorem 1.5. Then $f - g \in A(\overline{\Omega})$ and $g \in \mathcal{H}_\Phi$. This implies that $0 = \eta(g) = \eta(g) + \eta(f - g) = \eta(f)$. Therefore, $\eta = 0$, a contradiction.

The proof for $A(\overline{\Omega})$ follows similar steps, and is indeed even easier. □

We are now ready to give the proof of Theorem 1.1.

Proof of Theorem 1.1. We first show that $\mathcal{H}_\Phi = H^\infty(\Omega)$. By Lemma 1.2, \mathcal{H}_Φ is a closed unital subalgebra of $H^\infty(\Omega)$ of finite codimension. Let us assume by way of contradiction that

$\mathcal{H}_\Phi \neq H^\infty(\Omega)$. Gorin proves in [22] that every proper subalgebra of a commutative complex algebra is contained in a subalgebra of codimension one. Therefore, there exists a closed unital subalgebra A of $H^\infty(\Omega)$ of codimension one such that $\mathcal{H}_\Phi \subset A$.

We use the classification of the closed unital subalgebras of codimension one in a Banach algebra, which also is given in [22]. According to this classification, A must have one of the following two possible forms:

- (a) $A = \{f \in H^\infty : \psi_1(f) = \psi_2(f)\}$, for some $\psi_1, \psi_2 \in \mathfrak{M}(H^\infty(\Omega))$, $\psi_1 \neq \psi_2$.
- (b) $A = \ker \eta$, where $\eta \neq 0$ is a derivation at some $\psi \in \mathfrak{M}(H^\infty(\Omega))$.

We show that each of these two cases leads to a contradiction. In the case (a), Lemma 2.2 shows that $\psi_1(\mathbf{z}) \neq \psi_2(\mathbf{z}) \in \Omega$, yet $\Phi(\psi_1(\mathbf{z})) = \Phi(\psi_2(\mathbf{z}))$. Since Φ is injective, we get a contradiction.

In the case (b), Lemma 2.3 shows that $\psi(\mathbf{z}) \in \Omega$ and $\Phi'(\psi(\mathbf{z})) = 0$. This is a contradiction, because Φ' does not vanish in Ω .

The proof of the equality $\mathcal{A}_\Phi = A(\overline{\Omega})$ is identical. □

A few comments about the proof of Theorem 1.1 are in order. The first is about the classification of the derivations of $H^\infty(\Omega)$. We treat the case $\Omega = \mathbb{D}$, since the case of a finitely connected domain Ω is similar. We have already described the derivations at points $\psi \in \mathfrak{M}(H^\infty(\mathbb{D}))$ such that $\psi(\mathbf{z}) \in \mathbb{D}$ and given some properties about those derivations such that $\psi(\mathbf{z}) \in \mathbb{T}$. The first question is whether there exists any such (non-zero) derivations “supported on \mathbb{T} ”, and whether the case of a derivation η such that $\eta(\mathbf{z}) = 0$ but $\eta \neq 0$ that appeared in the proof of the theorem can really happen.

It is important to remark the existence of analytic disks inside each of the fibers of $\mathfrak{M}(H^\infty(\mathbb{D}))$ that project into \mathbb{T} under the map $\psi \mapsto \psi(\mathbf{z})$ (once again, we refer the reader to [29]). Thus there are maps of the form $\Psi : \mathbb{D} \rightarrow \mathfrak{M}(H^\infty(\mathbb{D}))$ such that for every $\lambda \in \mathbb{D}$ the point $\Psi_\lambda \in \mathfrak{M}(H^\infty(\mathbb{D}))$ lies in the same fiber (i.e., $\Psi_\lambda(\mathbf{z})$ is constant in λ) and such that the map $f(\lambda) \mapsto \Psi_\lambda(f)$ is an algebra homomorphism of $H^\infty(\mathbb{D})$ onto $H^\infty(\mathbb{D})$. This map Ψ endows its image \mathcal{D} in $\mathfrak{M}(H^\infty(\mathbb{D}))$ with an analytic structure. The complex derivative according to the analytic structure of \mathcal{D} gives a (non-zero) derivation at each of the points in \mathcal{D} (explicitly, these are maps $f \mapsto (d/d\lambda)|_{\lambda=\lambda_0} \Psi_{\lambda_0}(f)$). Clearly, if η is one of these derivations, then $\eta(\mathbf{z}) = 0$, because \mathbf{z} is constant on each of the fibers over \mathbb{T} , and hence in \mathcal{D} . However, we do not know whether these derivations are (up to a constant multiple) the only ones that exist over points of \mathcal{D} , or whether there exist (non-zero) derivations on points which do not belong to such analytic disks. It seems that there is not much information about the classification of the derivations of $H^\infty(\mathbb{D})$ in the literature.

Another comment is that one could use the results of Section 5 to simplify somewhat the proof of the Theorem 1.1. If we know that the algebra A is weak*-closed, then we only need to consider weak*-continuous complex homomorphisms and derivations.

3. Some lemmas about weakly singular integral operators

DEFINITION. We say that a domain $\Omega \subset \mathbb{C}$ satisfies the inner chord-arc condition if there is a constant $C > 0$ depending only on Ω such that for every $\zeta, z \in \overline{\Omega}$ there is a piecewise smooth curve $\gamma(\zeta, z)$ which joins ζ and z , is contained in Ω except for its endpoints, and whose length is smaller or equal than $C|\zeta - z|$.

LEMMA 3.1. *Let $U \subset \mathbb{C}$ be a domain satisfying the inner chord-arc condition, $\varphi \in A(\overline{U})$ with φ' of class Hölder α , $0 < \alpha \leq 1$ in U (so that φ' extends to \overline{U} by continuity). Let $K \subset \overline{U}$ be compact and $\Omega \subset U$ be a domain. Assume that $\varphi(\zeta) \neq \varphi(z)$ if $\zeta \in K$ and $z \in \overline{\Omega} \setminus \{\zeta\}$, and that φ' does not vanish in K .*

Then, the function

$$G(\zeta, z) = \frac{\varphi'(\zeta)}{\varphi(\zeta) - \varphi(z)} - \frac{1}{\zeta - z}$$

satisfies

$$|G(\zeta, z)| \leq C|\zeta - z|^{\alpha-1}, \quad \zeta \in K, z \in \overline{\Omega} \setminus \{\zeta\}.$$

Proof. Let us first check that

$$\left| \frac{\varphi(\zeta) - \varphi(z)}{\zeta - z} \right| \geq C_1 > 0, \quad \zeta \in K, z \in \overline{\Omega} \setminus \{\zeta\}. \quad (3.1)$$

To see this, put

$$h(\zeta, z) = \begin{cases} \frac{\varphi(\zeta) - \varphi(z)}{\zeta - z}, & \text{if } \zeta \in K, z \in \overline{\Omega} \setminus \{\zeta\}, \\ \varphi'(\zeta), & \text{if } \zeta = z \in K. \end{cases}$$

Since h is continuous on the compact set $K \times \overline{\Omega}$ and does not vanish, we get $|h| \geq C_1 > 0$, which implies (3.1).

If $\zeta \in K$ and $z \in \overline{\Omega} \setminus \{\zeta\}$, let $\gamma(\zeta, z) \subset U$ be an arc joining ζ and z and whose length is comparable to $|\zeta - z|$. Then

$$\begin{aligned} |\varphi(z) - \varphi(\zeta) - \varphi'(\zeta)(z - \zeta)| &= \left| \int_{\gamma(\zeta, z)} (\varphi'(u) - \varphi'(\zeta)) du \right| \leq C_2 \int_{\gamma(\zeta, z)} |u - \zeta|^\alpha du \\ &\leq C_3 |z - \zeta|^{\alpha+1}. \end{aligned} \quad (3.2)$$

Using (3.1) and (3.2), we get with $C = C_3/C_1$,

$$|G(\zeta, z)| = \frac{|\varphi(z) - \varphi(\zeta) - \varphi'(\zeta)(z - \zeta)|}{|\varphi(\zeta) - \varphi(z)||\zeta - z|} \leq C|\zeta - z|^{\alpha-1},$$

which proves the lemma. \square

The following lemma on the compactness of weakly singular integral operators may be well known to specialists. It appears throughout the literature in different forms. The one given here is similar to that in [32, Theorem 2.22], and it can be proved in the same way. Hence, we omit the proof.

LEMMA 3.2. *Let $\Omega \subset \mathbb{C}$ be bounded domain, $K \subset \mathbb{C}$ a compact piecewise smooth curve, and $G(\zeta, z)$ continuous in $(K \times \overline{\Omega}) \setminus \{(\zeta, \zeta) : \zeta \in K\}$ with $|G(\zeta, z)| \leq C|\zeta - z|^{-\beta}$ for some $\beta < 1$ and every $\zeta \in K, z \in \overline{\Omega} \setminus \{\zeta\}$.*

Then the operator

$$(T\psi)(z) = \int_K G(\zeta, z)\psi(\zeta)d\zeta \quad (3.3)$$

defines a compact operator $T : L^\infty(K) \rightarrow \mathcal{C}(\overline{\Omega})$.

If $\Gamma \subset \mathbb{C}$ is a piecewise smooth closed Jordan arc $\psi \in L^\infty(\Gamma)$, and φ and φ' are defined and continuous in Γ , we define the modified Cauchy integral

$$\mathcal{C}_\Gamma^\varphi(\psi)(z) = \int_\Gamma \frac{\varphi'(\zeta)}{\varphi(\zeta) - z} \psi(\zeta) d\zeta.$$

The function $\mathcal{C}_\Gamma^\varphi(\psi)$ is analytic in $\mathbb{C} \setminus \overline{\varphi(\Gamma)}$. We write $\mathcal{C}_\Gamma(\psi)$ for the usual Cauchy transform (i.e., when $\varphi(z) = z$).

4. Proof of Theorem 1.5

The following two lemmas are used in the proof of Theorem 1.5.

LEMMA 4.1. *Under the hypotheses of Theorem 1.5, if Γ is a piecewise smooth closed arc contained in $\overline{\Omega}_k$, then the operator defined by*

$$\psi \mapsto \mathcal{C}_\Gamma^{\varphi_k}(\psi) \circ \varphi_k - \mathcal{C}_\Gamma(\psi) \quad (4.1)$$

maps $L^\infty(\Gamma)$ into $A(\overline{\Omega})$ and is compact.

Proof. We compute

$$\mathcal{C}_\Gamma^{\varphi_k}(\psi) \circ \varphi_k - \mathcal{C}_\Gamma(\psi) = \int_\Gamma \left[\frac{\varphi'_k(\zeta)}{\varphi_k(\zeta) - \varphi_k(z)} - \frac{1}{\zeta - z} \right] \psi(\zeta) d\zeta. \quad (4.2)$$

Using Lemma 3.1 with $U = \Omega_k$, we have

$$\left| \frac{\varphi'_k(\zeta)}{\varphi_k(\zeta) - \varphi_k(z)} - \frac{1}{\zeta - z} \right| \leq C|\zeta - z|^{\alpha-1}, \quad \zeta \in \Gamma, \quad z \in \overline{\Omega} \setminus \{\zeta\}. \quad (4.3)$$

By Lemma 3.2, we see that the operator defined by (4.1) is compact from $L^\infty(\Gamma)$ to $\mathcal{C}(\overline{\Omega})$. Since its image clearly consists of analytic functions, the Lemma follows. \square

LEMMA 4.2. *Under the hypotheses of Theorem 1.5, let \widehat{J}_k be a closed arc contained the interior of J_k relative to $\partial\Omega$. If $\psi \in L^\infty(\widehat{J}_k)$ and $\mathcal{C}_{\widehat{J}_k}(\psi) \in H^\infty(\mathbb{C} \setminus \widehat{J}_k)$, then the modified Cauchy integral $\mathcal{C}_{\widehat{J}_k}^{\varphi_k}(\psi)$ belongs to $H^\infty(\mathbb{C} \setminus \varphi_k(\widehat{J}_k))$.*

Proof. We must verify that $\mathcal{C}_{\widehat{J}_k}^{\varphi_k}(\psi)$ is bounded in $\mathbb{C} \setminus \varphi_k(\widehat{J}_k)$. It is enough to check that it is bounded in $\varphi_k(\Omega_k) \setminus \varphi_k(\widehat{J}_k)$ as $\varphi_k(\Omega_k)$ is an open set containing $\varphi_k(\widehat{J}_k)$.

By Lemma 4.1, the function

$$\mathcal{C}_{\widehat{J}_k}^{\varphi_k}(\psi)(\varphi_k(z)) - \mathcal{C}_{\widehat{J}_k}(\psi)(z)$$

continues to a function in $A(\overline{\Omega}_k)$. In particular, it is bounded in $\Omega_k \setminus \widehat{J}_k$. Since $\mathcal{C}_{\widehat{J}_k}(\psi)$ is bounded in $\mathbb{C} \setminus \widehat{J}_k$, it follows that $\mathcal{C}_{\widehat{J}_k}^{\varphi_k}(\psi) \circ \varphi_k$ is bounded in $\Omega_k \setminus \widehat{J}_k$, or equivalently $\mathcal{C}_{\widehat{J}_k}^{\varphi_k}(\psi)$ is bounded in $\varphi_k(\Omega_k \setminus \widehat{J}_k)$. Since $\varphi_k(\Omega_k) \setminus \varphi_k(\widehat{J}_k) \subset \varphi_k(\Omega_k \setminus \widehat{J}_k)$, we conclude that $\mathcal{C}_{\widehat{J}_k}^{\varphi_k}(\psi) \in H^\infty(\mathbb{C} \setminus \varphi_k(\widehat{J}_k))$. \square

Proof of Theorem 1.5. Let us first justify that it is enough to prove the theorem for the case when each of the sets J_k is a single arc and these arcs intersect only at their endpoints. Write $J_k = \Gamma_{k,1} \cup \dots \cup \Gamma_{k,r_k}$, where $\Gamma_{k,j}$ are disjoint arcs, and put $\psi_{k,j} = \varphi_k$, for $j = 1, \dots, r_k$. Now pass to smaller arcs $\widetilde{\Gamma}_{k,j} \subset \Gamma_{k,j}$ such that the arcs $\widetilde{\Gamma}_{k,j}$ intersect only at endpoints but still cover all $\partial\Omega$. The functions $\psi_{k,j}$ and sets $\Gamma_{k,j}$ form an admissible family. Assume that the conclusion of Theorem 1.5 is true for this family, and let $F_{k,j}$ be the linear operators associated to each of the functions $\psi_{k,j}$. Putting $F_k = F_{k,1} + \dots + F_{k,r_k}$ and recalling that $\psi_{k,j} = \varphi_k$, we see that the conclusion of Theorem 1.5 is true for the family $\{\varphi_k\}$ as well.

We give the proof for a simply connected domain Ω . This case has the advantage that $\partial\Omega$ is a single Jordan curve, so the notation for numbering the arcs $J_k \subset \partial\Omega$ is easier. The proof for a multiply connected domain Ω is the essentially the same, except that the notation for the arcs J_k is a bit more complex.

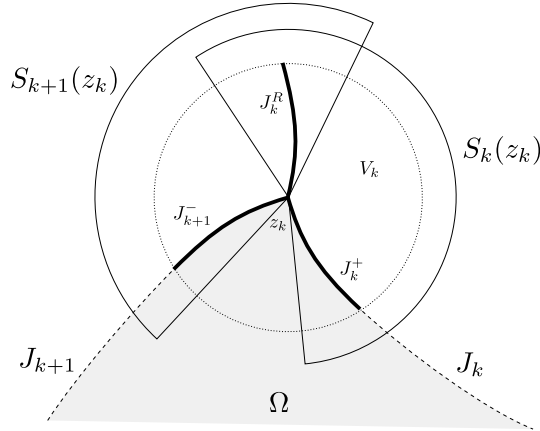


FIGURE 2. Geometric picture of the proof of Theorem 1.5

Let us assume that the arcs J_1, \dots, J_n are numbered in a cyclic order, i.e., in such a way that J_k intersects J_{k-1} and J_{k+1} (here and henceforth we consider subindices modulo n). Let $z_k \in \partial\Omega$ be the common endpoint of J_k and J_{k+1} , $k = 1, \dots, n$.

Let V_k be a small disk centered at z_k (its radius is determined later). Choose functions $\eta_1, \dots, \eta_n, \nu_1, \dots, \nu_n \in \mathcal{C}^\infty(\partial\Omega)$ such that $0 \leq \eta_k \leq 1$, $0 \leq \nu_k \leq 1$ on $\partial\Omega$, $\eta_1 + \dots + \eta_n + \nu_1 + \dots + \nu_n = 1$, $\text{supp } \nu_k \subset V_k \cap \partial\Omega$, and η_k is supported on the interior of J_k relative to $\partial\Omega$.

Put $J_k^+ = J_k \cap V_k$, $J_k^- = J_k \cap V_{k-1}$ and let R_k be a rigid rotation around the point z_k such that $J_k^R \stackrel{\text{def}}{=} R_k J_k^+$ is contained in $[(S_k(z_k) \cap S_{k+1}(z_k)) \setminus \overline{\Omega}] \cup \{z_k\}$ (see condition (d) in the definition of an admissible function). Figure 2 is a picture of the relevant geometric objects.

For $f \in H^\infty(\Omega)$, define

$$F_k(f) = \mathcal{C}_{J_k}^{\varphi_k}(f) - \mathcal{C}_{J_k^R}^{\varphi_k}((\nu_k f) \circ R_k^{-1}) + \mathcal{C}_{J_{k-1}^R}^{\varphi_k}((\nu_{k-1} f) \circ R_{k-1}^{-1}). \quad (4.4)$$

Let us first check that $F_k(f) \in H^\infty(\mathbb{D})$. To do this, put

$$\begin{aligned} G_k^+(f) &= \mathcal{C}_{J_k^+}^{\varphi_k}(\nu_k f) - \mathcal{C}_{J_k^R}^{\varphi_k}((\nu_k f) \circ R_k^{-1}), \\ G_k^-(f) &= \mathcal{C}_{J_{k+1}^-}^{\varphi_{k+1}}(\nu_k f) + \mathcal{C}_{J_k^R}^{\varphi_{k+1}}((\nu_k f) \circ R_k^{-1}). \end{aligned} \quad (4.5)$$

Then we can write $F_k(f)$ as

$$F_k(f) = \mathcal{C}_{J_k}^{\varphi_k}(\eta_k f) + G_k^+(f) + G_{k-1}^-(f),$$

because

$$\mathcal{C}_{J_k}^{\varphi_k}(f) = \mathcal{C}_{J_k}^{\varphi_k}(\eta_k f) + \mathcal{C}_{J_k^+}^{\varphi_k}(\nu_k f) + \mathcal{C}_{J_k^-}^{\varphi_k}(\nu_{k-1} f).$$

Since $f \in H^\infty(\Omega)$, and η_k is supported on a closed arc contained in the interior of J_k , it is easy to see that $\mathcal{C}_{J_k}(\eta_k f)$ belongs to $H^\infty(\mathbb{C} \setminus J_k)$. Lemma 4.2 allows us to conclude that $\mathcal{C}_{J_k}^{\varphi_k}(\eta_k f)$ belongs to $H^\infty(\mathbb{C} \setminus \varphi_k(J_k))$. As $\varphi_k(J_k) \subset \mathbb{T}$, this implies that $\mathcal{C}_{J_k}^{\varphi_k}(\eta_k f) \in H^\infty(\mathbb{D})$.

Since $|\varphi_k| < 1$ in Ω and $|\varphi_k| = 1$ in J_k , by the Schwarz reflection principle we can assume that

$$|\varphi_k| > 1 \text{ in } \Omega_k \setminus \Omega$$

just by making Ω_k smaller if necessary (i.e., replacing Ω_k by $U_k \cap \Omega_k$, where U_k is some open set containing $J_k \cup \Omega$).

The following claim is justified below.

CLAIM 1. $G_k^+(f) \in H^\infty(\mathbb{C} \setminus \varphi_k(J_k^+ \cup J_k^R))$ and $G_k^-(f) \in H^\infty(\mathbb{C} \setminus \varphi_{k+1}(J_{k+1}^- \cup J_k^R))$.

Since $|\varphi_k| = 1$ in J_k , $\varphi_k(J_k^+ \cup J_k^R) \cap \mathbb{D} = \emptyset$, and so by the claim, $G_k^+(f)$ and $G_k^-(f)$ belong to $H^\infty(\Omega)$. It follows that $F_k(f) \in H^\infty(\mathbb{D})$ for every $f \in H^\infty(\Omega)$. Moreover, it is clear from the proof of these lemmas that F_k maps $H^\infty(\Omega)$ into $H^\infty(\mathbb{D})$ and is bounded.

We next show that the linear map

$$f \mapsto f - \sum_{k=1}^n F_k(f) \circ \varphi_k,$$

is a compact operator on $H^\infty(\Omega)$, whose range is contained in $A(\overline{\Omega})$. A simple calculation using

$$f = \sum_{k=1}^n \mathcal{C}_{J_k}(f)$$

gives

$$f - \sum_{k=1}^n F_k(f) \circ \varphi_k = \sum_{k=1}^n A_k(f) + B_k(\nu_k f), \quad (4.6)$$

where

$$\begin{aligned} A_k(\psi) &= \mathcal{C}_{J_k}(\psi) - \mathcal{C}_{J_k}^{\varphi_k}(\psi) \circ \varphi_k, \\ B_k(\psi) &= \mathcal{C}_{J_k}^{\varphi_{k+1}}(\psi \circ R_k^{-1}) \circ \varphi_{k+1} - \mathcal{C}_{J_k}^{\varphi_k}(\psi \circ R_k^{-1}) \circ \varphi_k. \end{aligned} \quad (4.7)$$

By Lemma 4.1, the operator A_k is compact from $L^\infty(\partial\Omega)$ into $A(\overline{\Omega})$. To see that B_k has the same property, write

$$B_k(\psi) = [\mathcal{C}_{J_k}^{\varphi_{k+1}}(\psi \circ R_k^{-1}) \circ \varphi_{k+1} - \mathcal{C}_{J_k}^{\varphi_k}(\psi \circ R_k^{-1})] + [\mathcal{C}_{J_k}^{\varphi_k}(\psi \circ R_k^{-1}) - \mathcal{C}_{J_k}^{\varphi_k}(\psi \circ R_k^{-1}) \circ \varphi_k],$$

and apply Lemma 4.1 to each of the two terms in brackets.

It remains to prove that the operators F_k map $A(\overline{\Omega})$ into $A(\overline{\mathbb{D}})$. It is enough to check that if f is analytic on some open neighborhood of $\overline{\Omega}$, then $F_k(f) \in \mathcal{C}(\overline{\mathbb{D}})$, as the space of functions analytic on $\overline{\Omega}$ is dense in $A(\overline{\Omega})$ and F_k is bounded.

By (4.4) and properties of the modified Cauchy integral, $F_k(f)$ is continuous on $\overline{\mathbb{D}} \setminus \varphi_k(J_k)$. Next check that $F_k(f)$ extends by continuity to $\varphi_k(J_k)$. Since φ'_k does not vanish on J_k , there exists a continuous local inverse of φ_k on each point of $\varphi_k(J_k)$. This implies that it is enough to verify that $F_k(f) \circ \varphi_k$ is continuous in $\overline{\Omega}$. Put

$$\begin{aligned} \tilde{F}_k(f) &= \mathcal{C}_{J_k}(\eta_k f) + \tilde{G}_k^+(f) + \tilde{G}_{k-1}^-(f), \\ \tilde{G}_k^+(f) &= \mathcal{C}_{J_k^+}(\nu_k f) - \mathcal{C}_{J_k^R}((\nu_k f) \circ R_k^{-1}), \\ \tilde{G}_k^-(f) &= \mathcal{C}_{J_{k+1}^-}(\nu_k f) + \mathcal{C}_{J_k^R}((\nu_k f) \circ R_k^{-1}), \end{aligned}$$

i.e., replace the modified Cauchy integrals in the formulas for F_k , G_k^- and G_k^+ by regular Cauchy integrals to get \tilde{F}_k , \tilde{G}_k^- and \tilde{G}_k^+ . Arguing as above for the operators A_k and B_k , we see that $f \mapsto F_k(f) \circ \varphi_k - \tilde{F}_k(f)$ defines a compact operator whose range is contained in $\mathcal{C}(\overline{\Omega})$. Thus it is enough to show that $\tilde{F}_k(f) \in \mathcal{C}(\overline{\Omega})$.

By Lemma 4.3 below, it is easy to see that $\mathcal{C}_{J_k}(\eta_k f) \in \mathcal{C}(\overline{\Omega})$. We have

$$\tilde{G}_k^+(f) + \tilde{G}_k^-(f) = \mathcal{C}_{\partial\Omega \cap V_k}(\nu_k f).$$

Also by Lemma 4.3, the right hand side of this equality belongs to $\mathcal{C}(\overline{\Omega})$. Therefore, it suffices to check that $\tilde{G}_k^-(f) \in \mathcal{C}(\overline{\Omega})$.

Now $\tilde{G}_k^-(f) = \mathcal{C}_{J_{k+1}^- \cup J_k^R}(f)$, where $\tilde{f}(z) = (\nu_k f)(z)$ for $z \in J_{k+1}^-$, and $\tilde{f}(z) = (\nu_k f)(R_k^{-1}(z))$ for $z \in J_k^R$. Since f is analytic in a neighborhood of $\overline{\Omega}$, \tilde{f} is Lipschitz in $J_{k-1}^R \cup J_k^R$, and since \tilde{f}

vanishes identically near the endpoints of $J_{k-1}^- \cup J_k^R$, Lemma 4.3 implies that $\tilde{G}_k^-(f) \in \mathcal{C}(\overline{\Omega})$. This finishes the proof of the theorem. \square

Proof of Claim 1. We use the same techniques as those used in [25] to prove Theorem 4.1 to show that $g_k^- \stackrel{\text{def}}{=} G_k^-(f) \in H^\infty(\mathbb{C} \setminus \varphi_{k+1}(J_{k+1}^- \cup J_k^R))$. Similar reasoning can be applied to $G_k^+(f)$.

Let

$$h_k^+ = \mathcal{C}_{J_k^+}(\nu_k f), \quad h_k^- = \mathcal{C}_{J_{k+1}^-}(\nu_k f),$$

so that $h_k^- + h_k^+ = \mathcal{C}_{V_k \cap \partial\Omega}(\nu_k f)$, which because $f \in H^\infty(\Omega)$, belongs to $H^\infty(\mathbb{C} \setminus (V_k \cap \partial\Omega))$. Theorem 4.1 in [25] applies, and so $h_k^- + h_k^+ \circ R_k^{-1}$ belongs to $H^\infty(\mathbb{C} \setminus (J_k^- \cup J_k^R))$.

We next prove that g_k^- is bounded in $\mathbb{C} \setminus \varphi_{k+1}(J_{k+1}^- \cup J_k^R)$. It is clearly analytic in this set. Let S_{k+1}^- be an open circular sector with vertex on z_k , such that $J_{k+1}^- \cup J_k^R \subset S_{k+1}^- \cup \{z_k\}$ and $S_{k+1}^- \subset \Omega_{k+1}$. This circular sector can be chosen by shrinking one of the circular sectors which appear in condition (d) in the definition of an admissible family. We first show that g_k^- is bounded in $\varphi_{k+1}(S_{k+1}^- \setminus (J_{k+1}^- \cup J_k^R))$. To do this, observe that by a change of variables in the integral defining the Cauchy transform,

$$h_k^+ \circ R_k^{-1} = \mathcal{C}_{J_k^R}((\nu_k f) \circ R_k^{-1}).$$

Now compute

$$\begin{aligned} g_k^- \circ \varphi_{k+1} - (h_k^- + h_k^+ \circ R_k^{-1}) &= [\mathcal{C}_{J_{k+1}^-}^{\varphi_{k+1}}(\nu_k f) \circ \varphi_{k+1} - \mathcal{C}_{J_{k+1}^-}(\nu_k f)] \\ &\quad + [\mathcal{C}_{J_k^R}^{\varphi_{k+1}}((\nu_k f) \circ R_k^{-1}) \circ \varphi_{k+1} - \mathcal{C}_{J_k^R}((\nu_k f) \circ R_k^{-1})]. \end{aligned}$$

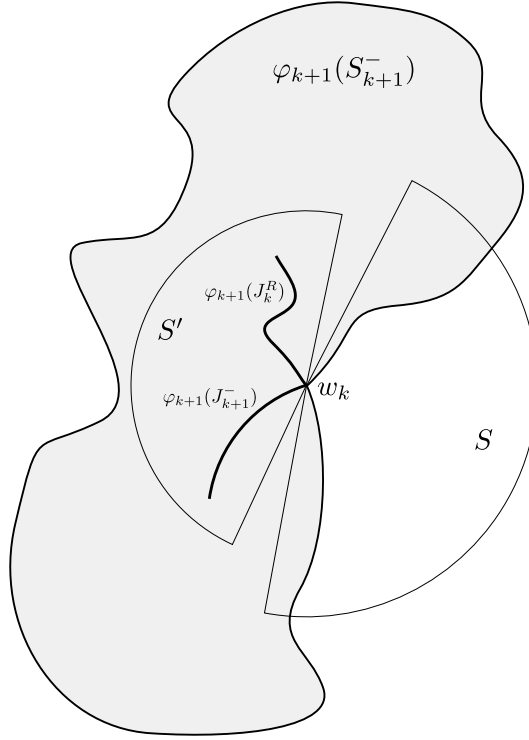
A similar argument to the one used in Lemma 4.2 shows that each of the expressions in square brackets is bounded in $S_{k+1}^- \setminus (J_{k+1}^- \cup J_k^R)$. Therefore, $g_k^- \circ \varphi_{k+1}$ is also bounded in this set as $h_k^- + h_k^+ \circ R_k^{-1}$ is bounded there. It follows that g_k^- is bounded in $\varphi_{k+1}(S_{k+1}^- \setminus (J_{k+1}^- \cup J_k^R))$.

It remains to prove that g_k^- is bounded in $\mathbb{C} \setminus \varphi_{k+1}(S_{k+1}^-)$. If S is an open circular sector, we say that its straight edges are the two line segments which form a part of its boundary. Put $w_k = \varphi_{k+1}(z_k)$. Choose two open circular sectors S and S' with vertex w_k having the following properties (see Figure 3):

- $\overline{S} \cap \overline{S'} = \{w_k\}$.
- $\varphi_{k+1}(J_{k+1}^- \cup J_k^R) \subset S' \cup \{w_k\}$.
- $\mathbb{D}_\varepsilon(w_k) \setminus \varphi_{k+1}(S_{k+1}^-) \subset S \cup \{w_k\}$ for some $\varepsilon > 0$.
- The straight edges of S are contained in $\varphi_{k+1}(S_{k+1}^-) \cup \{w_k\}$.

Such circular sectors can be chosen by shrinking V_k if necessary, using the fact that φ_{k+1} is conformal at z_k .

It is enough to show that g_k^- is bounded in S , because $g_k^-(z)$ is clearly uniformly bounded when z is away from $\varphi_{k+1}(J_{k+1}^- \cup J_k^R)$, and we have already seen that g_k^- is bounded in $\varphi_{k+1}(S_{k+1}^- \setminus (J_{k+1}^- \cup J_k^R))$. This is done by using a weak form of the Phragmén-Lindelöf principle, in the same manner as in [25]. Since g_k^- is bounded in the straight edges of S except at the vertex w_k (the straight edges are contained in $\varphi_{k+1}(S_{k+1}^- \setminus (J_{k+1}^- \cup J_k^R))$, except for w_k), it suffices to show that g_k^- is $O(|z - w_k|^{-1})$ as $z \rightarrow w_k$, $z \in S$.

FIGURE 3. The circular sectors S and S'

First, estimate

$$\begin{aligned}
 |(z - w_k)g_k^-(z)| &\leq \int_{J_{k+1}^-} \frac{|z - w_k|}{|\varphi_{k+1}(\zeta) - z|} |(\nu_k f)(\zeta) \varphi'_{k+1}(\zeta)| |d\zeta| \\
 &\quad + \int_{J_k^R} \frac{|z - w_k|}{|\varphi_{k+1}(\zeta) - z|} |(\nu_k f)(\zeta) \varphi'_{k+1}(\zeta)| |d\zeta| \\
 &\leq \|f\|_\infty \int_{J_{k+1}^- \cup J_k^R} \frac{|z - w_k|}{|\varphi_{k+1}(\zeta) - z|} |\varphi'_{k+1}(\zeta)| |d\zeta|.
 \end{aligned}$$

We claim that $a(\zeta, z) \stackrel{\text{def}}{=} |z - w_k|/|\varphi_{k+1}(\zeta) - z|$ is uniformly bounded for $z \in S$ and $\zeta \in J_k^- \cup J_k^R$, which follows from the observation that $\varphi_{k+1}(\zeta) \in S'$ and $z \in S$, so that $a(\zeta, z) \leq C$ due to the geometry of the cones S and S' . The last integral is therefore uniformly bounded, so g_k^- is $O(|z - w_k|^{-1})$ and we conclude that g_k^- belongs to $H^\infty(\mathbb{C} \setminus \varphi_{k+1}(J_{k+1} \cup J_k^R))$.

To see that g_k^+ belongs to $H^\infty(\mathbb{C} \setminus \varphi_k(J_k^+ \cup J_k^R))$, use similar reasoning with $h_k^+ - h_k^+ \circ R_k^{-1}$ instead of $h_k^- + h_k^+ \circ R_k^{-1}$, an appropriate circular sector S_k^+ for S_{k+1}^- , and φ_k in place of φ_{k+1} . \square

The following lemma is well known from the classical theory of Cauchy integrals. See, for instance, [20, Chapter I, Section 5.1].

LEMMA 4.3. *Let Γ be a piecewise smooth Jordan curve and Ω the region interior to it. If ψ is of class Hölder α on Γ , $0 < \alpha < 1$, then $\mathcal{C}_\Gamma(\psi)$ is of class Hölder α in $\overline{\Omega}$.*

5. Weak* closedness

In this section we prove that the algebra \mathcal{H}_Φ is weak*-closed in H^∞ . First recall a well known result about weak*-continuity of adjoint operators, the proof of which is elementary and so is omitted.

LEMMA 5.1. *If X is a Banach space and $T : X \rightarrow X$ is a bounded operator, then its adjoint $T^* : X^* \rightarrow X^*$ is continuous in the weak*-topology of X^* .*

The operator T is called the predual of T^* . Thus any operator with a predual is weak*-continuous, a condition applying to many integral operators on L^∞ .

LEMMA 5.2. *Let $T : L^\infty(\partial\Omega) \rightarrow L^\infty(\partial\Omega)$ be defined by*

$$(Tf)(z) = \int_{\partial\Omega} G(\zeta, z)f(\zeta) d\zeta,$$

where $G : \partial\Omega \times \partial\Omega \rightarrow \mathbb{C}$ is a measurable function satisfying

$$\int_{\partial\Omega} |G(\zeta, z)| |d\zeta| \leq C$$

for every $z \in \partial\Omega$. Then the operator S defined by

$$(Sg)(\zeta) = \int_{\partial\Omega} G(\zeta, z)g(z) dz$$

is a bounded operator $S : L^1(\partial\Omega) \rightarrow L^1(\partial\Omega)$ and satisfies $S^* = T$.

Proof. Fubini's Theorem shows that

$$\left| \int_{\partial\Omega} (Sg)(\zeta)f(\zeta) d\zeta \right| \leq C \|g\|_1 \|f\|_\infty,$$

and so S is bounded on $L^1(\partial\Omega)$. Another application of Fubini's Theorem gives $S^* = T$. \square

Recall from (4.6) and the proof of Theorem 1.5 that the operator

$$K(f) = f - \sum_{k=1}^n F_k(f) \circ \varphi_k,$$

is a weakly singular integral operator of the form

$$K(f)(z) \mapsto \int_{\partial\Omega} G(\zeta, z)f(\zeta) d\zeta,$$

where the function G is continuous outside the diagonal $\{\zeta = z\}$ and $|G(\zeta, z)| \leq C|\zeta - z|^{-\beta}$, for some $\beta < 1$. Also, for each $\zeta \in \partial\Omega$, $G(\zeta, z)$ is analytic in $z \in \Omega$. Thus, K is compact from $L^\infty(\Omega)$ to $H^\infty(\Omega)$. By Lemma 5.2, the operator K has a predual, so is weak*-continuous.

LEMMA 5.3. *For every $\varepsilon > 0$, there is an operator $K_\varepsilon : L^\infty(\partial\Omega) \rightarrow L^\infty(\partial\Omega)$ of finite rank which has a predual and such that $\|K_\varepsilon - K\| < \varepsilon$.*

Proof. Fix $\varepsilon > 0$. Since G is continuous outside $\{\zeta = z\}$, there exist $\alpha_j \in L^\infty(\partial\Omega)$ and $\beta_j \in L^1(\partial\Omega)$ such that

$$\int_{\partial\Omega} \left| G(\zeta, z) - \sum_{j=1}^N \alpha_j(z) \beta_j(\zeta) \right| d\zeta < \varepsilon/2$$

for every $z \in \partial\Omega$. This implies that the finite rank operator K_ε defined by

$$K_\varepsilon(\psi)(z) = \int_{\partial\Omega} \sum_{j=1}^N \alpha_j(z) \beta_j(\zeta) \psi(\zeta) d\zeta$$

satisfies $\|K_\varepsilon - K\| \leq \varepsilon/2$. Clearly, by Lemma 5.2, the operator K_ε has a predual. \square

LEMMA 5.4. *Let $L(f) = \sum_{k=1}^n F_k(f) \circ \varphi_k$ be the operator $L : H^\infty(\Omega) \rightarrow H^\infty(\Omega)$ of Theorem 1.5. Then the range of L is weak*-closed in $H^\infty(\Omega)$.*

Proof. We have $L = (I - K)|_{H^\infty(\Omega)}$. By the preceding lemma with $\varepsilon = 1$, there is a finite rank operator K_1 such that $\|K_1 - K\| < 1$. Put $M = H^\infty(\Omega) + K_1(L^\infty(\partial\Omega))$. Then, since $H^\infty(\Omega)$ is weak*-closed in $L^\infty(\partial\Omega)$, M is a weak*-closed subset of $L^\infty(\Omega)$ such that $H^\infty(\Omega)$ has finite codimension in M . Define $\Delta = I - (K - K_1)$. Note that $K_1(L^\infty(\partial\Omega)) \subset M$ and $K(L^\infty(\partial\Omega)) \subset H^\infty(\Omega) \subset M$. Since

$$\Delta^{-1} = \sum_{j=0}^{\infty} (K - K_1)^j,$$

this series being convergent in operator norm, we also have $\Delta^{-1}M \subset M$.

Now observe that

$$L(H^\infty(\Omega)) = (I - K)H^\infty(\Omega) = \Delta(I - \Delta^{-1}K_1)H^\infty(\Omega).$$

Put $X = (I - \Delta^{-1}K_1)H^\infty(\Omega)$ and note that $\ker K_1 \cap H^\infty(\Omega) \subset X$. Since $\ker K_1 \cap H^\infty(\Omega)$ is weak*-closed and has finite codimension in M , and $X \subset M$, it follows that X is weak* closed.

It remains to show that ΔX is weak*-closed. It is enough to check that Δ^{-1} is weak*-continuous. Since K and K_1 have preduals, it follows that Δ has a predual. Therefore, Δ^{-1} also has a predual, and so it is weak*-continuous. \square

Finally, we can show that \mathcal{H}_Φ is weak*-closed in $H^\infty(\Omega)$. The argument is similar to the proof of Lemma 1.2.

LEMMA 5.5. *If $\Phi : \overline{\Omega} \rightarrow \overline{\mathbb{D}}^n$ is admissible, then \mathcal{H}_Φ is weak*-closed in $H^\infty(\Omega)$.*

Proof. We have already seen in the proof of Lemma 1.2 that the range of the operator $f \mapsto \sum F_k(f) \circ \varphi_k$, $f \in H^\infty(\Omega)$, has finite codimension in $H^\infty(\Omega)$. By the preceding lemma, the range is also weak*-closed. Since \mathcal{H}_Φ contains this range, we get that \mathcal{H}_Φ is weak*-closed. \square

6. Glued subalgebras

In this section we characterize the maximal ideal space and the derivations of finite codimensional subalgebras of a unital commutative Banach algebra. The arguments used are purely algebraic and similar results hold for arbitrary unital commutative complex algebras.

In the algebraic setting, one should replace the maximal ideal space by the set of all (unital) homomorphisms of the algebra into the complex field and disregard every reference made to the topology, such as closed subspaces and continuity of homomorphisms and derivations.

Let A be a commutative unital Banach algebra. A *glued subalgebra* of A is understood to be a (unital) subalgebra of the form

$$B = \{f \in A : \alpha_j(f) = \beta_j(f), j = 1, \dots, r\}, \quad (6.1)$$

where $\alpha_j, \beta_j \in \mathfrak{M}(A)$ and $\alpha_j \neq \beta_j$ for $j = 1, \dots, r$. We define the set of points of A glued in B as

$$G(A, B) = \{\alpha_j : j = 1, \dots, r\} \cup \{\beta_j : j = 1, \dots, r\} \subset \mathfrak{M}(A).$$

Our first goal is to characterize the space $\mathfrak{M}(B)$ in terms of $\mathfrak{M}(A)$. Since B is a subalgebra of A , there is a map $i^* : \mathfrak{M}(A) \rightarrow \mathfrak{M}(B)$ which sends each complex homomorphism $\psi \in \mathfrak{M}(A)$ to its restriction $\psi|_B \in \mathfrak{M}(B)$. We first show that i^* is onto. To do this, we need to use the so called “lying over lemma”, which applies to integral ring extensions.

Recall that if R is a subring of some ring S , then S is called *integral* over R if for every $\alpha \in S$ there is a monic polynomial $p \in R[x]$ such that $p(\alpha) = 0$. It is well known that if B is a finite codimensional subalgebra of some algebra A , then A is integral over B .

The following “lying over lemma” or Cohen-Seidenberg theorem is a standard result from commutative algebra. It was originally proved in [14].

LEMMA 6.1 (Lying over lemma). *If S is integral over R and P is a prime ideal in R , then there is a prime ideal Q in S such that $P = Q \cap R$ (we say that Q is lying over P). If Q is a prime ideal in S lying over P , then Q is maximal if and only if P is maximal.*

LEMMA 6.2. *If B is a finite codimensional closed unital subalgebra of a commutative unital Banach algebra A , then $i^* : \mathfrak{M}(A) \rightarrow \mathfrak{M}(B)$ is onto.*

Proof. Let $\psi_B \in \mathfrak{M}(B)$ and put $P = \ker \psi_B$. Then P is a maximal ideal in B . By the lying over lemma, there is a maximal ideal Q in A such that $Q \cap B = P$. Since P is closed and has finite codimension in A and $Q \supset P$, it follows that Q is closed. Every maximal ideal in a Banach algebra has codimension one, so Q has codimension one in A . Therefore, there exists a unique $\psi_A \in \mathfrak{M}(A)$ such that $\ker \psi_A = Q$. The equality $Q \cap B = P$ implies $i^*(\psi_A) = \psi_B$. \square

Note that since $\mathfrak{M}(A)$ is compact and i^* is continuous, it follows that i^* is topologically a quotient map.

In the purely algebraic setting, it is no longer true that every maximal ideal has codimension one. However, one can still show that Q has codimension one in A . Indeed, note that P has finite codimension in B . Therefore P has finite codimension in A . Since $P \subset Q$, it follows that Q has finite codimension in A . Now, A/Q is a field, because Q is a maximal ideal in A . Also, A/Q is a finite dimensional vector space over \mathbb{C} . Since finite field extensions are algebraic and \mathbb{C} is algebraically closed, it follows that A/Q is isomorphic to \mathbb{C} , so that Q has codimension one in A .

The following is a kind of “interpolation” lemma. It will be very useful in the sequel [18].

LEMMA 6.3. *Let A be a commutative unital Banach algebra. If ψ_0, \dots, ψ_s are distinct points in $\mathfrak{M}(A)$, then there is some $f \in A$ such that $\psi_j(f) = 0$ for $j = 1, \dots, s$, but $\psi_0(f) = 1$.*

Proof. Fix $j \in \{1, \dots, s\}$. There is some $f_j \in A$ such that $\psi_0(f_j) \neq 0$ and $\psi_j(f_j) = 0$, for if this were not the case, then $\ker \psi_0 \subset \ker \psi_j$, which would imply that $\ker \psi_0 = \ker \psi_j$ since both kernels have codimension one in A . Hence we would have $\psi_0 = \psi_j$, a contradiction.

For f_1, \dots, f_s chosen in this way,

$$f = \prod_{j=1}^s \frac{f_j}{\psi_0(f_j)}.$$

has the required properties. \square

LEMMA 6.4. *If B is a glued subalgebra of A and $\psi_B \in \mathfrak{M}(B)$, then either $(i^*)^{-1}(\{\psi_B\}) \subset G(A, B)$ or $(i^*)^{-1}(\{\psi_B\}) = \{\psi_A\}$, for some $\psi_A \notin G(A, B)$.*

Proof. Assume that we have distinct $\psi_A, \tilde{\psi}_A \in (i^*)^{-1}(\{\psi_B\})$ with $\psi_A \notin G(A, B)$. By Lemma 6.3, there is an $f \in A$ such that $\psi_A(f) = 1$ and $\psi(f) = 0$ for $\psi \in G(A, B) \cup \{\tilde{\psi}_A\}$, as $\psi_A \notin G(A, B) \cup \{\tilde{\psi}_A\}$ by hypothesis. Then $f \in B$, because $\alpha_j(f) = \beta_j(f) = 0$, for $j = 1, \dots, r$, and

$$1 = \psi_A(f) = \psi_B(f) = \tilde{\psi}_A(f) = 0,$$

because $\psi_A|_B = \psi_B = \tilde{\psi}_A|_B$. This is a contradiction. \square

We next describe the derivations of B in terms of the derivations of A . This requires the following well-known characterization of derivations: *A linear functional η on A is a derivation at $\psi \in \mathfrak{M}(A)$ if and only if $\eta(1) = 0$ and $\eta(fg) = 0$ whenever $f, g \in A$ and $\psi(f) = \psi(g) = 0$.*

LEMMA 6.5. *Let B be a glued subalgebra of A , and η_B a derivation of B at a point $\psi_B \in \mathfrak{M}(B)$. Put*

$$\{\psi_A^1, \dots, \psi_A^s\} = (i^*)^{-1}(\{\psi_B\}) \subset \mathfrak{M}(A)$$

(this set is finite by Lemma 6.4). Then there exist unique derivations $\eta_A^1, \dots, \eta_A^s$ of A at the points $\psi_A^1, \dots, \psi_A^s$ respectively such that

$$\eta_B = (\eta_A^1 + \dots + \eta_A^s)|_B.$$

Proof. For each $k = 1, \dots, s$, use Lemma 6.3 to obtain an $f_k \in A$ such that $\psi_A^j(f_k) = \delta_{jk}$ and $\psi(f_k) = 0$ for $\psi \in G(A, B) \setminus \{\psi_A^1, \dots, \psi_A^s\}$. Put $g_k = 2f_k - f_k^2$. Then g_k also satisfies $\psi_A^j(g_k) = \delta_{jk}$ and $\psi(g_k) = 0$ for $\psi \in G(A, B) \setminus \{\psi_A^1, \dots, \psi_A^s\}$.

Define η_A^j by

$$\eta_A^j(f) = \eta_B(g_j^2(f - \psi_A^j(f))), \quad f \in A.$$

Since $\psi(g_j^2(f - \psi_A^j(f))) = 0$ for every $\psi \in G(A, B)$, $g_j^2(f - \psi_A^j(f)) \in B$ and so η_A^j is well defined.

We claim that η_A^j is a derivation of A at ψ_A^j . It is clearly linear and $\eta_A^j(1) = 0$. Take $f, g \in A$ with $\psi_A^j(f) = \psi_A^j(g) = 0$. Then,

$$\eta_A^j(fg) = \eta_B(g_j^2 fg) = 0,$$

because $g_j f$ and $g_j g$ belong both to B and $\psi_B(g_j f) = \psi_B(g_j g) = 0$ as $\psi(g_j f) = \psi(g_j g) = 0$ for every $\psi \in G(A, B) \cup \{\psi_A^1, \dots, \psi_A^s\}$. It follows that η_A^j is a derivation of A at ψ_A^j .

Now we check that $(\eta_A^1 + \dots + \eta_A^s)|_B = \eta_B$. For this, put

$$g_0 = g_1^2 + \dots + g_s^2.$$

Note that $\psi_A^1(g_0) = \cdots = \psi_A^s(g_0) = 1$ and $\psi(g_0) = 0$ for $\psi \in G(A, B) \setminus \{\psi_A^1, \dots, \psi_A^s\}$. If α_j, β_j are as in (6.1), then $i^*(\alpha_j) = i^*(\beta_j)$. Therefore, either α_j, β_j both belong to $(i^*)^{-1}(\{\psi_B\}) = \{\psi_A^1, \dots, \psi_A^s\}$ or they both belong to $G(A, B) \setminus \{\psi_A^1, \dots, \psi_A^s\}$. Hence, $\alpha_j(g_0) = \beta_j(g_0)$, because $\alpha_j(g_0)$ and $\beta_j(g_0)$ are both 1 or both 0. Therefore, $g_0 \in B$. Also, $\psi_B(g_0) = \psi_A^1(g_0) = 1$.

Take any $f \in B$. Then

$$\sum_{j=1}^s \eta_A^j(f) = \sum_{j=1}^s \eta_B(g_j^2(f - \psi_A^j(f))) = \eta_B(g_0(f - \psi_B(f))) = \eta_B(f - \psi_B(f)) = \eta_B(f),$$

because $\psi_B(f - \psi_B(f)) = 0$, and $\psi_B(g_0) = 1$. This shows that $(\eta_A^1 + \cdots + \eta_A^s)|B = \eta_B$.

To prove uniqueness, assume that $\tilde{\eta}_A^1, \dots, \tilde{\eta}_A^s$ are derivations of A at $\psi_A^1, \dots, \psi_A^s$ respectively and such that $(\tilde{\eta}_A^1 + \cdots + \tilde{\eta}_A^s)|B = \eta_B$.

Since $\psi_A^j(g_j) = \psi_A^j(f_j) = 1$,

$$\tilde{\eta}_A^j(g_j^2) = 2\tilde{\eta}_A^j(g_j)\psi_A^j(g_j) = 2\tilde{\eta}_A^j(g_j) = 2\tilde{\eta}_A^j(2f_j - f_j^2) = 4\tilde{\eta}_A^j(f_j) - 4\tilde{\eta}_A^j(f_j)\psi_A^j(f_j) = 0,$$

and so $\tilde{\eta}_A^j(g_j^2) = 0$. Thus for any $f \in A$,

$$\tilde{\eta}_A^j(f) = \tilde{\eta}_A^j(g_j^2 f) = \tilde{\eta}_A^j(g_j^2(f - \psi_A^j(f))),$$

and if $j \neq k$, then

$$\tilde{\eta}_A^k(g_j^2(f - \psi_A^j(f))) = 0,$$

because $\psi_A^k(g_j) = \psi_A^k(g_j(f - \psi_A^j(f))) = 0$. Also, since $\psi(g_j^2(f - \psi_j(f))) = 0$ for every $\psi \in G(A, B)$, $g_j^2(f - \psi_j(f)) \in B$. Hence,

$$\tilde{\eta}_A^j(f) = \sum_{k=1}^s \tilde{\eta}_A^k(g_j^2(f - \psi_A^j(f))) = \eta_B(g_j^2(f - \psi_A^j(f))) = \eta_A^j(f).$$

This shows that $\tilde{\eta}_A^j = \eta_A^j$ for $j = 1, \dots, s$. □

REMARK. If we denote by $\text{Der}_\psi(A)$ the linear space of derivations of A at $\psi \in \mathfrak{M}(A)$, then the lemma above shows that

$$\text{Der}_{\psi_B}(B) \cong \bigoplus_{\psi \in (i^*)^{-1}(\{\psi_B\})} \text{Der}_\psi(A).$$

7. The proofs of Theorems 1.3 and 1.4

This section is devoted to proving Theorems 1.3 and 1.4. We use results by Gamelin [21] on finite codimensional subalgebras of uniform algebras. In particular, we need to use his concept of a θ -subalgebra, which we define here in the context of unital commutative Banach algebras. If A is a Banach algebra and $\theta \in \mathfrak{M}(A)$, a θ -subalgebra of A is a subalgebra H such that there is a chain $H = H_0 \subset H_1 \subset \cdots \subset H_n = A$ where H_k is the kernel of some derivation in H_{k+1} at the point θ (the restriction map $i^* : \mathfrak{M}(H_{k+1}) \rightarrow \mathfrak{M}(H_k)$ is a bijection, so the maximal ideal spaces of all the algebras H_k can be viewed as being the same).

The main result of [21] concerning finite codimensional subalgebras is that, roughly speaking, every such subalgebra can be constructed by first passing to a glued subalgebra and then taking the intersection of a finite number of θ_j -subalgebras of the glued subalgebra.

LEMMA 7.1. Assume that $\Phi : \overline{\Omega} \rightarrow \overline{\mathbb{D}}^n$ is admissible. There is a finite set $X \subset \Omega$ and $N \in \mathbb{N}$ such that if $f \in H^\infty(\Omega)$ has a zero of order N at each point of X , then $f \in \mathcal{H}_\Phi$. The same is true if one replaces $H^\infty(\Omega)$ by $A(\overline{\Omega})$ and \mathcal{H}_Φ by \mathcal{A}_Φ .

Proof. According to [21, Theorem 9.8], there is a glued subalgebra H_0 of $H^\infty(\Omega)$ and a finite family of θ_j -subalgebras H_j of H_0 such that

$$\mathcal{H}_\Phi = H_1 \cap \cdots \cap H_r.$$

(Here $\theta_j \in \mathfrak{M}(H_0)$.)

Put $G = G(H^\infty(\Omega), H_0)$. By Lemma 2.2, $\psi(\mathbf{z}) \in \Omega$ for every $\psi \in G$. Consider the map $i^* : \mathfrak{M}(H^\infty(\Omega)) \rightarrow \mathfrak{M}(H_0)$ and put $Y = (i^*)^{-1}(\{\theta_1, \dots, \theta_r\})$. By Lemma 6.4, Y is a finite set.

If $\psi \in Y$, then $\psi(\mathbf{z}) \in \Omega$, since either $\psi \in G$ or ψ is the unique preimage of some θ_j . In the latter case, since $\mathcal{H}_\Phi \subset H_j$, there is some derivation η of H_0 at θ_j such that $\eta|_{\mathcal{H}_\Phi} = 0$ but $\eta \neq 0$. By Lemma 6.5, η extends to a derivation $\hat{\eta}$ of $H^\infty(\Omega)$ at ψ . By Lemma 2.3, $\psi(\mathbf{z}) \in \Omega$, because $\hat{\eta}|_{\mathcal{H}_\Phi} = 0$.

We claim that $X = \{\psi(\mathbf{z}) : \psi \in G \cup Y\} \subset \Omega$ is the desired set. Note that X is finite. Also, if $f \in H^\infty(\Omega)$ vanishes on X then $f \in H_0$, because $\psi(f) = 0$ for every $\psi \in G$.

Let k_j be the codimension of H_j in H_0 . Assume that $f \in H^\infty(\Omega)$ has a zero of order 2^{k_j} at every point of X . Then f can be factored as $f = f_1 \cdots f_{2^{k_j}}$, where each of the 2^{k_j} functions belongs to $H^\infty(\Omega)$ and vanishes on X . This implies $f \in H_j$ by [21, Lemma 9.3]. Thus, the lemma holds with $N = \max_j 2^{k_j}$.

The proof for $A(\overline{\Omega})$ is similar. \square

Proof of Theorem 1.3. We only show that $\Phi^* H^\infty(\mathcal{V}) = \mathcal{H}_\Phi$ as the proof of $\Phi^* A(\overline{\Omega}) = \mathcal{A}_\Phi$ is identical. The inclusion $\mathcal{H}_\Phi \subset \Phi^* H^\infty(\mathcal{V})$ follows from (1.1). We examine the reverse inclusion.

Let $n \in \mathbb{N}$ be the integer and $X = \{z_1, \dots, z_r\} \subset \Omega$ the finite set from Lemma 7.1. Put $w_j = \Phi(z_j) \in \mathcal{V}$, $j = 1, \dots, r$. Take $F \in H^\infty(\mathcal{V})$ and observe that F extends to an analytic function, which we also denote by F , on a neighborhood in \mathbb{C}^n of each of the points w_j , $j = 1, \dots, r$. Let $P \in \mathbb{C}[z_1, \dots, z_n]$ be a polynomial such that

$$D^\alpha P(z_j) = D^\alpha F(z_j), \quad 0 \leq |\alpha| \leq N, \quad j = 1, \dots, r.$$

Then $\Phi^*(F - P) = F \circ \Phi - P \circ \Phi$ has a zero of order N at each point of X , so $\Phi^*(F - P) \in \mathcal{H}_\Phi$. Also, $\Phi^*P \in \mathcal{H}_\Phi$, because it is a polynomial in $\varphi_1, \dots, \varphi_n$. It follows that $\Phi^*F \in \mathcal{H}_\Phi$. \square

Proof of Theorem 1.4. By Lemma 1.2, \mathcal{H}_Φ is a closed subspace of $H^\infty(\Omega)$. Define an operator $L : \mathcal{H}_\Phi \rightarrow \mathcal{H}_\Phi$ by $Lf = \sum_{k=1}^n F_k(f) \circ \varphi_k$, where F_1, \dots, F_n are the operators that appear in Theorem 1.5. Since $I - L$ is compact, by the Fredholm theory there are bounded operators $R, P : \mathcal{H}_\Phi \rightarrow \mathcal{H}_\Phi$ such that P has finite rank and $I = LR + P$. The operator P can be written as

$$Pf = \sum_{k=1}^r \alpha_k(f) g_k, \quad f \in H^\infty(\Omega),$$

for some $g_k \in \mathcal{H}_\Phi$ and $\alpha_k \in \mathcal{H}_\Phi^*$. The functions g_k can be expressed as

$$g_k = \sum_{j=1}^l \prod_{i=1}^n f_{j,i,k} \circ \varphi_i$$

(so as to have the same number of multiplicands and terms in these sums, we can take some of the $f_{j,i,k}$ equal to 0 or 1).

Take an $f \in H^\infty(\mathcal{V})$. By Theorem 1.3, $\Phi^*f \in \mathcal{H}_\Phi$. Put

$$F(z_1, \dots, z_n) = \sum_{k=1}^n F_k(R\Phi^*f)(z_k) + \sum_{k=1}^r \sum_{j=1}^l \alpha_k(\Phi^*f) \prod_{i=1}^n f_{j,i,k}(z_i).$$

Then $\Phi^*F = F \circ \Phi = LR\Phi^*f + P\Phi^*f = \Phi^*f$, so $F|_{\mathcal{V}} = f$.

If $g \in H^\infty(\mathbb{D})$, then $\|g(z_k)\|_{\mathcal{SA}(\mathbb{D}^n)} = \|g\|_\infty$, so

$$\begin{aligned} \|F\|_{\mathcal{SA}(\mathbb{D}^n)} &\leq \sum_{k=1}^n \|F_k(R\Phi^*f)\|_\infty + \sum_{k=1}^r \sum_{j=1}^l |\alpha_k(\Phi^*f)| \cdot \prod_{i=1}^n \|f_{j,i,k}\|_\infty \\ &\leq C\|\Phi^*f\|_{H^\infty(\mathbb{D})} = C\|f\|_{H^\infty(\mathcal{V})}. \end{aligned}$$

□

8. Continuous families of admissible functions

In this section we prove a lemma concerning a family of admissible functions $\{\Phi_\varepsilon\}$ which depends continuously on ε . The lemma shows that one can get operators F_k^ε as in Theorem 1.5 that satisfy certain continuity properties in ε . This result is used in an application to the study of complete K -spectral sets in our forthcoming article [18].

LEMMA 8.1. *Let $\Phi_\varepsilon = (\varphi_1^\varepsilon, \dots, \varphi_n^\varepsilon) : \overline{\Omega} \rightarrow \overline{\mathbb{D}}^n$, $0 \leq \varepsilon \leq \varepsilon_0$ be a collection of functions. Assume that Ψ_ε is admissible for every ε , and, moreover, that one can choose sets Ω_k in the definition of an admissible collection so as not to depend on ε . Assume that $\varphi_k^\varepsilon \in \mathcal{C}^{1+\alpha}(\Omega_k)$, with $0 < \alpha < 1$, and that the mapping $\varepsilon \mapsto \varphi_k^\varepsilon$ is continuous from $[0, \varepsilon_0]$ to $\mathcal{C}^{1+\alpha}(\Omega_k)$.*

Then there exist bounded linear operators $F_k^\varepsilon : A(\overline{\Omega}) \rightarrow A(\overline{\mathbb{D}})$, such that for

$$L_\varepsilon(f) = \sum F_k^\varepsilon(f) \circ \varphi_k^\varepsilon,$$

and $0 \leq \varepsilon \leq \varepsilon_0$, $L_\varepsilon - I$ is a compact operator on $A(\overline{\Omega})$, the mapping $\varepsilon \mapsto L_\varepsilon$ is norm continuous, and $\|F_k^\varepsilon\| \leq C$ for $k = 1, \dots, n$, where C is a constant independent of k and ε .

The proof of Lemma 8.1 uses the following technical fact:

LEMMA 8.2. *Let Ω be a bounded domain satisfying the inner chord-arc condition, $\{\varphi_\varepsilon\}_{0 \leq \varepsilon \leq \varepsilon_0} \subset A(\overline{\Omega})$ with φ_ε' of class Hölder α in Ω and such that the mapping $\varepsilon \mapsto \varphi_\varepsilon$ is continuous from $[0, \varepsilon_0]$ to $\mathcal{C}^{1+\alpha}(\Omega)$. Let $K \subset \overline{\Omega}$ be compact. Assume that $\varphi_\varepsilon(\zeta) \neq \varphi_\varepsilon(z)$ if $\zeta \in K$, $z \in \overline{\Omega} \setminus \{\zeta\}$ and $0 \leq \varepsilon \leq \varepsilon_0$. Assume also that for each $0 \leq \varepsilon \leq 1$, φ_ε' does not vanish in K . Then for $\zeta \in K$, $z \in \overline{\Omega} \setminus \{\zeta\}$, and $\varepsilon, \delta \in [0, \varepsilon_0]$,*

$$\left| \frac{\varphi_\varepsilon'(\zeta)}{\varphi_\varepsilon(\zeta) - \varphi_\varepsilon(z)} - \frac{\varphi_\delta'(\zeta)}{\varphi_\delta(\zeta) - \varphi_\delta(z)} \right| \leq C \|\varphi_\varepsilon' - \varphi_\delta'\|_{\mathcal{C}^\alpha} |\zeta - z|^{\alpha-1}.$$

Proof. First check that

$$\left| \frac{\varphi_\varepsilon(\zeta) - \varphi_\varepsilon(z)}{\zeta - z} \right| \geq C > 0, \quad \zeta \in K, \quad z \in \overline{\Omega} \setminus \{\zeta\}, \quad 0 \leq \varepsilon \leq \varepsilon_0. \quad (8.1)$$

To see this, put

$$h(\zeta, z, \varepsilon) = \begin{cases} \frac{\varphi_\varepsilon(\zeta) - \varphi_\varepsilon(z)}{\zeta - z}, & \text{if } \zeta \in K, z \in \overline{\Omega} \setminus \{\zeta\}, \\ \varphi_\varepsilon'(\zeta), & \text{if } \zeta = z \in K, \end{cases}$$

which is continuous on the compact set $K \times \overline{\Omega} \times [0, \varepsilon_0]$. As h does not vanish, $|h| \geq C > 0$, implying (8.1).

Since $|\psi(u) - \psi(\zeta)| \leq \|\psi\|_{C^\alpha} |u - \zeta|^\alpha$ and $|\psi(u)| \leq \|\psi\|_{C^\alpha}$,

$$\begin{aligned} & |\varphi'_\varepsilon(\zeta)\varphi'_\delta(u) - \varphi'_\delta(\zeta)\varphi'_\varepsilon(u)| \\ &= \left| [\varphi'_\varepsilon(u) - \varphi'_\varepsilon(\zeta)][\varphi'_\delta(\zeta) - \varphi'_\varepsilon(\zeta)] + \varphi'_\varepsilon(\zeta)[\varphi'_\varepsilon(u) - \varphi'_\varepsilon(\zeta) + \varphi'_\delta(\zeta) - \varphi'_\delta(u)] \right| \\ &\leq \left| \varphi'_\varepsilon(u) - \varphi'_\varepsilon(\zeta) \right| \left| \varphi'_\delta(\zeta) - \varphi'_\varepsilon(\zeta) \right| + |\varphi'_\varepsilon(\zeta)| \left| (\varphi'_\varepsilon - \varphi'_\delta)(u) - (\varphi'_\varepsilon - \varphi'_\delta)(\zeta) \right| \\ &\leq \|\varphi'_\varepsilon\|_{C^\alpha} |u - \zeta|^\alpha \|\varphi'_\varepsilon - \varphi'_\delta\|_{C^\alpha} + \|\varphi'_\varepsilon\|_{C^\alpha} \|\varphi'_\varepsilon - \varphi'_\delta\|_{C^\alpha} |u - \zeta|^\alpha. \end{aligned}$$

But

$$\varphi'_\varepsilon(\zeta)[\varphi_\delta(\zeta) - \varphi_\delta(z)] - \varphi'_\delta(\zeta)[\varphi_\varepsilon(\zeta) - \varphi_\varepsilon(z)] = \int_{\gamma(z, \zeta)} (\varphi'_\varepsilon(\zeta)\varphi'_\delta(u) - \varphi'_\delta(\zeta)\varphi'_\varepsilon(u)) \, du,$$

and so

$$\begin{aligned} \left| \varphi'_\varepsilon(\zeta)[\varphi_\delta(\zeta) - \varphi_\delta(z)] - \varphi'_\delta(\zeta)[\varphi_\varepsilon(\zeta) - \varphi_\varepsilon(z)] \right| &\leq C \|\varphi'_\varepsilon - \varphi'_\delta\|_{C^\alpha} \int_{\gamma(z, \zeta)} |u - \zeta|^\alpha |du| \\ &\leq C \|\varphi'_\varepsilon - \varphi'_\delta\|_{C^\alpha} |z - \zeta|^{\alpha+1}. \end{aligned}$$

Combining this with (8.1),

$$\begin{aligned} \left| \frac{\varphi'_\varepsilon(\zeta)}{\varphi_\varepsilon(\zeta) - \varphi_\varepsilon(z)} - \frac{\varphi'_\delta(\zeta)}{\varphi_\delta(\zeta) - \varphi_\delta(z)} \right| &= \left| \frac{\varphi'_\varepsilon(\zeta)[\varphi_\delta(\zeta) - \varphi_\delta(z)] - \varphi'_\delta(\zeta)[\varphi_\varepsilon(\zeta) - \varphi_\varepsilon(z)]}{[\varphi_\varepsilon(\zeta) - \varphi_\varepsilon(z)][\varphi_\delta(\zeta) - \varphi_\delta(z)]} \right| \\ &\leq C \|\varphi'_\varepsilon - \varphi'_\delta\|_{C^\alpha} \frac{|z - \zeta|^{\alpha+1}}{|\varphi_\varepsilon(\zeta) - \varphi_\varepsilon(z)| |\varphi_\delta(\zeta) - \varphi_\delta(z)|} \\ &\leq C \|\varphi'_\varepsilon - \varphi'_\delta\|_{C^\alpha} |z - \zeta|^{\alpha-1}. \end{aligned}$$

□

Proof of Lemma 8.1. The construction of the functions η_k and ν_k used in the proof of Theorem 1.5 depends solely on the geometry of Ω , and not on the functions φ_k . So define F_k^ε by equation (4.4), replacing φ_k with φ_k^ε . Then $L_\varepsilon - I$ is compact by the proof of Theorem 1.5. We also define A_k^ε and B_k^ε by equation (4.7), with φ_k^ε instead of φ_k . Then by (4.6),

$$L_\varepsilon(f) - L_\delta(f) = \sum_{k=1}^n (A_k^\delta - A_k^\varepsilon)(f) + (B_k^\delta - B_k^\varepsilon)(\nu_k f).$$

Note that

$$(A_k^\delta - A_k^\varepsilon)(f)(z) = \mathcal{C}_{J_k}^{\varphi_k^\varepsilon}(f)(z) - \mathcal{C}_{J_k}^{\varphi_k^\delta}(f)(z) = \int_{J_k} \left(\frac{(\varphi_k^\varepsilon)'(\zeta)}{\varphi_k^\varepsilon(\zeta) - \varphi_k^\varepsilon(z)} - \frac{(\varphi_k^\delta)'(\zeta)}{\varphi_k^\delta(\zeta) - \varphi_k^\delta(z)} \right) f(\zeta) \, d\zeta.$$

Using Lemma 8.2 and the fact that

$$\int_{J_k} |z - \zeta|^{\alpha-1} |d\zeta| < \infty,$$

we have $\|A_k^\delta - A_k^\varepsilon\| \leq C \|(\varphi_k^\varepsilon)' - (\varphi_k^\delta)'\|_{C^\alpha}$. Also, $\|B_k^\delta - B_k^\varepsilon\| \leq C \|(\varphi_k^\varepsilon)' - (\varphi_k^\delta)'\|_{C^\alpha}$ by similar reasoning. These inequalities imply that L_ε depends continuously on ε in the norm topology.

To see that $\|F_k^\varepsilon\| \leq C$, with C independent of ε , one must examine the proofs of Theorem 1.5 and Lemma 4.2 to check that the constants that bound the operators which appear there can be taken to be independent of ε . First, we give the details concerning the proof of Lemma 4.2.

Instead of (4.3), we require the inequality

$$\left| \frac{(\varphi_k^\varepsilon)'(\zeta)}{\varphi_k^\varepsilon(\zeta) - \varphi_k^\varepsilon(z)} - \frac{1}{\zeta - z} \right| \leq C |\zeta - z|^{\alpha-1}, \quad \zeta \in J_k, \, z \in \overline{\Omega}_k \setminus \{\zeta\}, \, 0 \leq \varepsilon \leq \varepsilon_0. \quad (8.2)$$

Here C should be a constant independent of ε , so we cannot simply apply Lemma 3.1. To prove this inequality, apply Lemma 3.1 to φ_k^0 and get (8.2) for $\varepsilon = 0$, and then use Lemma 8.2 to

obtain

$$\left| \frac{(\varphi_k^\varepsilon)'(\zeta)}{\varphi_k^\varepsilon(\zeta) - \varphi_k^\varepsilon(z)} - \frac{(\varphi_k^0)'(\zeta)}{\varphi_k^0(\zeta) - \varphi_k^0(z)} \right| \leq C|\zeta - z|^{\alpha-1}, \quad \zeta \in J_k, \quad z \in \overline{\Omega}_k \setminus \{\zeta\}, \quad 0 \leq \varepsilon \leq \varepsilon_0,$$

where C is independent of ε . Then (8.2) follows from the triangle inequality.

Let \widehat{J}_k and ψ be as in the statement of Lemma 4.2. We verify that $\|\mathcal{C}_{\widehat{J}_k}^{\varphi_k^\varepsilon}(\psi)\|_\infty \leq C$, with C independent of ε . By the proof of Lemma 4.2, $\varphi_k^\varepsilon(\Omega_k)$ is an open set containing $\varphi_k^\varepsilon(\widehat{J}_k)$. Moreover, it follows from the continuity of φ_k^ε in ε and the compactness of the interval $[0, \varepsilon_0]$ that the distance from $\varphi_k^\varepsilon(\widehat{J}_k)$ to the boundary of $\varphi_k^\varepsilon(\Omega_k)$ is bounded below by a positive constant independent of ε . Therefore, it suffices to show that $\mathcal{C}_{\widehat{J}_k}^{\varphi_k^\varepsilon}(\psi)$ is bounded in $\varphi_k^\varepsilon(\Omega_k) \setminus \varphi_k^\varepsilon(\widehat{J}_k)$ by a constant independent of ε , because when the distance from some point z to $\varphi_k^\varepsilon(\widehat{J}_k)$ is greater than a constant, $\mathcal{C}_{\widehat{J}_k}^{\varphi_k^\varepsilon}(\psi)(z)$ is readily bounded by a constant independent of z and ε .

To show that $\mathcal{C}_{\widehat{J}_k}^{\varphi_k^\varepsilon}(\psi)$ is bounded in $\varphi_k^\varepsilon(\Omega_k) \setminus \varphi_k^\varepsilon(\widehat{J}_k)$, we prove as in Lemma 4.2 that $\mathcal{C}_{\widehat{J}_k}^{\varphi_k^\varepsilon}(\psi) \circ \varphi_k^\varepsilon$ is bounded in $\Omega_k \setminus \widehat{J}_k$. Write (4.2) for φ_k^ε instead of φ_k and \widehat{J}_k instead of Γ , and then use (8.2) to obtain

$$\left| \mathcal{C}_{\widehat{J}_k}^{\varphi_k^\varepsilon}(\psi)(\varphi_k^\varepsilon(z)) - \mathcal{C}_{\widehat{J}_k}^{\varphi_k^\varepsilon}(\psi)(z) \right| \leq C\|\psi\|_\infty \int_{\widehat{J}_k} |\zeta - z|^{\alpha-1} d\zeta \leq C\|\psi\|_\infty,$$

where C is independent of ε . Since $\mathcal{C}_{\widehat{J}_k}^{\varphi_k^\varepsilon}(\psi) \in H^\infty(\mathbb{C} \setminus \widehat{J}_k)$, we get the required bound.

It remains to check that the H^∞ norms in Claim 1 (see the proof of Theorem 1.5) can be bounded by a constant independent of ε . We can apply methods similar to the ones that we have used for Lemma 4.2. Define $(G_k^-)^\varepsilon$ as in (4.5), replacing φ_k with φ_k^ε . Put $g_k^\varepsilon = (G_k^-)^\varepsilon(f)$. This is in $H^\infty(\mathbb{C} \setminus \varphi_{k+1}^\varepsilon(J_{k+1}^- \cup J_k^R))$ by Claim 1. We want to show that g_k^ε is bounded by a constant independent of ε .

Define h_k^+ and h_k^- as in the proof of Claim 1 (these functions do not depend on φ_k). Compute

$$g_k^\varepsilon \circ \varphi_{k+1}^\varepsilon - (h_k^- + h_k^+ \circ R_k^{-1}) = [\mathcal{C}_{J_{k+1}^-}^{\varphi_{k+1}^\varepsilon}(\nu_k f) \circ \varphi_{k+1}^\varepsilon - \mathcal{C}_{J_{k+1}^-}(\nu_k f)] + [\mathcal{C}_{J_k^R}^{\varphi_{k+1}^\varepsilon}((\nu_k f) \circ R_k^{-1}) \circ \varphi_{k+1}^\varepsilon - \mathcal{C}_{J_k^R}((\nu_k f) \circ R_k^{-1})].$$

Arguing as before and using (8.2), each of the two terms in brackets is bounded by a constant independent of φ . Since $h_k^- + h_k^+ \circ R_k^{-1} \in H^\infty(\mathbb{C} \setminus (J_{k+1}^- \cup J_k^R))$, $g_k^\varepsilon \circ \varphi_{k+1}^\varepsilon$ is uniformly bounded in $S_{k+1}^- \setminus (J_{k+1}^- \cup J_k^R)$, and so g_k^ε is uniformly bounded in $\varphi_{k+1}^\varepsilon(S_{k+1}^- \setminus (J_{k+1}^- \cup J_k^R))$.

Now choose open circular sectors S_ε and S'_ε with vertex on $w_k^\varepsilon = \varphi_{k+1}^\varepsilon(z_k)$ such that $\overline{S_\varepsilon} \cap \overline{S'_\varepsilon} = \{w_k^\varepsilon\}$, and satisfying the following conditions (see Figure 3 in the proof of Theorem 1.5):

- $\varphi_{k+1}^\varepsilon(J_{k+1}^- \cup J_k^R) \subset S'_\varepsilon \cup \{w_k^\varepsilon\}$.
- $\mathbb{D}_{\varepsilon_0}(w_k^\varepsilon) \setminus \varphi_{k+1}^\varepsilon(S_{k+1}^-) \subset S_\varepsilon \cup \{w_k^\varepsilon\}$, for some $\varepsilon_0 > 0$.
- The straight edges of S_ε are contained in $\varphi_{k+1}^\varepsilon(S_{k+1}^-) \cup \{w_k^\varepsilon\}$.

This can be done by the continuity of $\varphi_{k+1}^\varepsilon$ in ε and by shrinking V_k if necessary.

To show that g_k^ε is bounded in S_ε , we use the Phragmén-Lindelöf principle as in the proof of Claim 1. There, we proved that g_k^ε is $O(|z - w_k^\varepsilon|^{-1})$ as $z \rightarrow w_k^\varepsilon$. Thus, g_k^ε is bounded in S_ε by the supremum of $|g_k^\varepsilon|$ on the straight edges of S_ε . Since these straight edges are contained in $\varphi_{k+1}^\varepsilon(S_{k+1}^-)$ and we had a bound for φ_k^ε which is uniform in ε on this set, there is a bound on S_ε which is also uniform in ε .

Clearly, $g_k^\varepsilon(z)$ is uniformly bounded in ε and z when the distance from z to $\varphi_{k+1}^\varepsilon(J_{k+1}^- \cup J_k^R)$ is greater than a positive constant. Also, g_k^ε is uniformly bounded on $U_\varepsilon \setminus \varphi_{k+1}^\varepsilon(J_{k+1}^- \cup J_k^R)$, where U_ε is some open set containing $\varphi_{k+1}^\varepsilon(J_{k+1}^- \cup J_k^R)$ and such that the distance from ∂U_ε to $\varphi_{k+1}^\varepsilon(J_{k+1}^- \cup J_k^R)$ is bounded below by a positive constant independent of ε . This finishes the proof. \square

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